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# Witt Rings and Brauer Groups under Multiquadratic Extensions. II

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## 0. INTRODUCTION

Let  $F$  be a field of characteristic not 2, and suppose  $M$  is a multiquadratic extension of  $F$ . That is,  $M/F$  is a finite abelian extension of exponent 2, so that  $M = F(\sqrt{G})$  for some finite subgroup  $G \subseteq \dot{F}/\dot{F}^2$ , where  $\dot{F} = F - \{0\}$ . In studying Brauer groups and products of quaternion algebras, the second author was led to consider the homology groups  $N_i(M/F)$  of a certain complex  $\mathcal{B}_{M/F}$  associated to the extension  $M/F$ . This complex appears in [11, (3.1); 30] and in (1.1) below.

The purpose of the present work is to investigate the first homology group  $N_1(M/F)$  and to exhibit its close connections with quadratic form theory. The higher homology groups are examined in [11, 30].

The field  $F$  is said to have property  $P_1(n)$  if  $N_1(M/F) = 1$  for every multiquadratic extension  $M/F$  with  $[M:F] \leq 2^n$ . Properties  $P_1(1)$  and  $P_1(2)$  always hold, but there are examples [29] of fields  $F$  for which  $P_1(3)$  fails. These examples are generalized in Section 5.

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One major motivation for this work is the relationship between the properties  $P_1(n)$  and certain questions concerning quadratic forms. We outline those ideas here, referring to Section 2 for more details. Suppose  $G$  is a given subgroup of  $\dot{F}/\dot{F}^2$ . The relation of  $G$ -equivalence of quadratic forms over  $F$  was introduced in [28] as a generalization of the chain-equivalence theorem of Witt. A weaker relation on quadratic forms is given by congruence mod  $\mathfrak{A}(G)$ , where  $\mathfrak{A}(G)$  is the ideal in the Witt ring  $WF$  generated by all the binary forms  $\langle 1, -g \rangle$  for  $g \in G$ . The group  $G$  is said to have binary piecewise equivalence (PE) if these two equivalence relations coincide for binary forms over  $F$ . We show that  $F$  satisfies property  $P_1(3)$  if and only if every subgroup  $G \subseteq \dot{F}/\dot{F}^2$  of order 4 has binary PE. The counterexamples to  $P_1(3)$  provide the first known examples of groups not satisfying PE.

Another connection is provided by the behavior of common values represented by two quadratic forms under field extensions. An extension  $E/F$  has the common value (CV) property if for any  $F$ -forms  $\alpha, \beta$  having  $D_E(\alpha) \cap D_E(\beta) \neq \emptyset$ , then in fact  $D_E(\alpha) \cap D_E(\beta) \cap \dot{E} \neq \emptyset$ . Here  $D_E(\alpha)$  is the set of values in  $\dot{E}$  represented by  $\alpha \otimes E$ . We prove that  $F$  satisfies  $P_1(3)$  if and only if every quadratic extension  $E/F$  satisfies CV. The counterexamples to  $P_1(3)$  provide the first known examples of quadratic extensions  $E/F$  for which CV fails. A third equivalent version of  $P_1(3)$  is given in terms of representational abstract Witt rings, in the sense of [17]. The counterexamples to  $P_1(3)$  obtained in Section 5 provide examples of fields  $F$  such that the abstract Witt ring  $WF/\langle\langle -a \rangle\rangle WF$  is not representational, for some  $a \in \dot{F}$ .

In Section 3 we examine some classes of fields which do satisfy property  $P_1$ , namely, the fields with Hasse number  $\tilde{u} \leq 2$ , and the global fields. The first part of Section 3 deals with the sequence  $\mathcal{C}_{M/F}$  formed from the rings of continuous functions on spaces of orderings. This sequence is always exact and there is a natural homomorphism of complexes  $\mathcal{B}_{M/F} \rightarrow \mathcal{C}_{M/F}$  induced by the signature maps. When  $\tilde{u}(F) \leq 2$ , the property  $P_1$  easily follows from these ideas.

In Section 4 we consider a certain intersection property  $I(n)$  of ideals in the Witt ring  $WF$ . In order to study this property, we introduce a complex  $\mathcal{F}_{M/F}^2$ , formed from ideals in  $WF$ , and examine the natural homomorphism  $\mathcal{F}_{M/F}^2 \rightarrow \mathcal{B}_{M/F}$ . If  $\tilde{u}(F) \leq 4$ , we show using this homomorphism that  $I(n)$  and  $P_1(n)$  are equivalent properties.

We exhibit in Section 5 a class of fields not satisfying property  $P_1(3)$ . For example, if  $F = L(x)$  where  $L$  is an algebraic number field, or if  $F = L(x, y)$  where  $L$  is any field of characteristic 0, then  $P_1(3)$  fails for  $F$ . In particular if  $F = \mathbb{C}(x, y)$ , where  $\mathbb{C}$  is the field of complex numbers, then  $P_1(3)$  fails for  $F$  and  $u(F) = 4$ . By the results of Section 4, this provides a counterexample to the property  $I(3)$  as well.

1. THE GROUPS  $N_1(M/F)$  AND PROPERTY  $P_1(n)$ 

For a multiquadratic field extension  $M/F$ , we describe the associated chain complex  $\mathcal{B}_{M/F}$  and its first homology group  $N_1(M/F)$ , following [29]. The higher homology groups are treated in [11, 29, 30]. In this section we show that  $N_1(M/F) = 1$  whenever  $[M:F] = 4$  and compute  $N_1(M/F)$  as a quotient of norm groups when  $[M:F] = 8$ . Some associated properties  $P_1(n)$ ,  $R(n)$  and  $Q(n)$  are also discussed.

Let  $F$  be a field of characteristic not 2, and  $\dot{F} = F - \{0\}$ . A *multiquadratic extension*  $M/F$  is a finite abelian extension of exponent 2 (a finite 2-Kummer extension). Equivalently,  $M = F(\sqrt{G})$  where  $G = G(M/F)$  is the kernel of the natural map  $\dot{F}/\dot{F}^2 \rightarrow \dot{M}/\dot{M}^2$ . If  $\{a_1, \dots, a_n\}$  generate  $G$  in  $\dot{F}/\dot{F}^2$ , then  $M = F(\sqrt{a_1}, \dots, \sqrt{a_n})$ . Here, as usual we abuse the notation by using the same letter for an element of  $\dot{F}$  and its class in  $\dot{F}/\dot{F}^2$ .

The complex  $\mathcal{B}_{M/F}$  associated to this extension is the following:

$$\mathcal{B}_{M/F}: \bigoplus_{L \in Q(M/F)} (\dot{L}/\dot{L}^2) \xrightarrow{\varphi} G \otimes (\dot{F}/\dot{F}^2) \xrightarrow{\psi} \text{Br}_2(F) \xrightarrow{r} \text{Br}_2(M) \quad (1.1)$$

This is a shortened version of the complex considered in [11, 30]. The notations used in (1.1) will now be defined.

The tensor product is taken over  $\mathbb{F}_2$  (the field of 2 elements), where  $\dot{F}/\dot{F}^2$  is viewed as an  $\mathbb{F}_2$ -vector space and  $G$  as a subspace. Although the “addition” in this space is written multiplicatively, we write the elements of the tensor product additively. A typical element of  $G \otimes (\dot{F}/\dot{F}^2)$  is written as  $\sum_{g \in G} g \otimes x_g$ , for some  $x_g \in \dot{F}$ . Since 1 is the “zero” for the space  $\dot{F}/\dot{F}^2$ , we have  $1 \otimes x = 0$ , for all  $x \in \dot{F}$ .

$\text{Br}_2(F)$  is the group of elements of order 1 or 2 in the Brauer group  $\text{Br}(F)$ . The map  $r$  is the natural map given by extending scalars. Define  $\text{Br}_2(M/F) = \ker r$ . If  $a, b \in \dot{F}$ , then  $[a, b] = [a, b]_F \in \text{Br}_2(F)$  denotes the class of the quaternion algebra  $(a, b/F)$ . This “quaternion symbol” induces a homomorphism  $\psi: (\dot{F}/\dot{F}^2) \otimes (\dot{F}/\dot{F}^2) \rightarrow \text{Br}_2(F)$ . The map  $\psi = \psi_M$  in (1.1) is the restriction of this quaternion map. So if  $x_g \in \dot{F}$  is given for each  $g \in G$ , then

$$\psi \left( \sum_{g \in G} g \otimes x_g \right) = \prod_{g \in G} [g, x_g].$$

Define  $\text{Dec}(M/F) = \text{im } \psi$ . Clearly  $\text{im } \psi \subseteq \ker r$ , that is,  $\text{Dec}(M/F) \subseteq \text{Br}_2(M/F)$ . The elements of  $\text{Dec}(M/F)$  are the algebras in  $\text{Br}_2(M/F)$  which “decompose” nicely with respect to the extension  $M/F$ .

Define  $Q(M/F)$  to be the set of all quadratic extensions of  $F$  contained in  $M$ . Then for  $M = F(\sqrt{G})$  we have  $L \in Q(M/F)$  iff  $L = F(\sqrt{\kappa(L)})$  for some

unique element  $\kappa(L) \in G$ ,  $\kappa(L) \neq 1$ . The map  $\varphi = \varphi_M$  in (1.1) is defined as follows:

$$\text{if } (z_L) \in \bigoplus_{L \in Q(M/F)} (\dot{L}/\dot{L}^2), \text{ then } \varphi((z_L)) = \sum_{L \in Q(M/F)} \kappa(L) \otimes N_{L/F}(z_L).$$

For any  $z_L \in \dot{L}$ , the norm  $N_{L/F}(z_L)$  in  $\dot{F}$  is a value represented by the quadratic form  $\langle 1, -\kappa(L) \rangle$ , so that  $[\kappa(L), N_{L/F}(z_L)] = 1$  in  $\text{Br}_2(F)$ . Therefore  $\text{im } \varphi \subseteq \ker \psi$ . We consider the homology groups of the complex  $\mathcal{B}_{M/F}$ .

**1.2 DEFINITION.**  $N_1(M/F)$  and  $N_2(M/F)$  denote the homology groups of  $\mathcal{B}_{M/F}$  at  $G \otimes (\dot{F}/\dot{F}^2)$  and at  $\text{Br}_2(F)$ , respectively.

Then

$$N_2(M/F) = \frac{\ker r}{\text{im } \psi} = \frac{\text{Br}_2(M/F)}{\text{Dec}(M/F)}.$$

This notation agrees with that of [11, Sect. 3].

Let  $i = 1$  or  $2$  and let  $n > 0$  be an integer. We say, as in [11, 29, 30] that  $F$  has *Property*  $P_i(n)$  if  $N_i(M/F) = 1$  for every multiquadratic extension  $M/F$  with  $[M:F] \leq 2^n$ . If  $F$  satisfies  $P_i(n)$  for all  $n$ , then we say  $F$  has *Property*  $P_i$ . In this paper we concentrate on the properties  $P_1(n)$ .

**1.3. Remark.** We note here that if  $F$  satisfies  $P_1$ , then the quaternion symbol homomorphism  $g_F: k_2 F \rightarrow \text{Br}_2(F)$  is injective. This homomorphism is described in [7, 24] and is shown to be equivalent to the Witt invariant map  $\bar{c}_F: I^2 F / I^3 F \rightarrow \text{Br}_2(F)$ . Now suppose  $F$  satisfies  $P_1$ , so that every sequence  $\mathcal{B}_{M/F}$  is exact at the first place. Taking direct limits we conclude that the sequence

$$\mathcal{B}_F: \bigoplus (\dot{L}/\dot{L}^2) \xrightarrow{\varphi} (\dot{F}/\dot{F}^2) \otimes (\dot{F}/\dot{F}^2) \xrightarrow{\psi} \text{Br}_2(F)$$

is exact. Here the sum is taken over all quadratic extensions  $L$  of  $F$ . It is not hard to check that  $k_2 F \cong \text{coker } \varphi$  and that  $\psi$  induces the map  $g_F$ . Hence the injectivity of  $g_F$  is equivalent to the exactness of  $\mathcal{B}_F$ . Compare [11, (1.5); 30, (0.1)].

In working with the homology group  $N_1(M/F)$ , it is convenient to modify the first term of the complex  $\mathcal{B}(M/F)$ . If  $L \in Q(M/F)$ , then  $L = F(\sqrt{a})$ , where  $a = \kappa(L) \in G$ . For any  $a \in \dot{F} - \dot{F}^2$ , define  $N_F(a)$  to be the set of all nonzero norms from the field  $F(\sqrt{a})$ . If  $a \in \dot{F}^2$ , set  $N_F(a) = \dot{F}$ . Consider the map

$$\hat{\varphi}: \bigoplus_{g \in G} N_F(g) \rightarrow G \otimes (\dot{F}/\dot{F}^2)$$

defined by sending  $(x_g)_{g \in G} \mapsto \sum_{g \in G} g \otimes x_g$ . It is easy to check that  $\text{im } \varphi = \text{im } \hat{\varphi}$ . (The  $a = 1$  term contributes nothing, since  $1 \otimes x = 0$ .) Therefore, in calculating  $N_1(M/F)$  we can use this map  $\hat{\varphi}$  in place of  $\varphi$ .

The norm groups  $N_F(a)$  appear often in this work. Note that  $N_F(a) = D_F(\langle 1, -a \rangle)$ , the set of values in  $\dot{F}$  represented by the quadratic form  $\langle 1, -a \rangle$ . Another description is

$$N_F(a) = \{x \in \dot{F} \mid [a, x] = 1\}.$$

The complex  $\mathcal{B}_{M/F}$  was defined using the group  $\dot{L}/\dot{L}^2$  in order to emphasize the duality between the first and last terms of the extended version of the complex [11, (3.1)], as well as to motivate the analogous complexes described in Sections 3 and 4. For the remainder of this section we will use the map  $\hat{\varphi}$  rather than  $\varphi$ .

In order to state  $P_1(n)$  more concretely, suppose the group  $G \subseteq \dot{F}/\dot{F}^2$  has an  $\mathbb{F}_2$ -basis  $\{a_1, \dots, a_m\}$ . For each subset  $P \subseteq \{1, 2, \dots, m\}$  define  $a_P = \prod_{i \in P} a_i$ . (By convention,  $a_\emptyset = 1$ .) Then  $G = \{a_P \mid P \subseteq \{1, \dots, m\}\}$ . Any element of  $G \otimes (\dot{F}/\dot{F}^2)$  can be written uniquely as  $\sum_{i=1}^m a_i \otimes x_i$ , for some  $x_i \in \dot{F}$ .

1.4. LEMMA. *The property  $P_1(n)$  can be stated as follows: For every group  $G \subseteq \dot{F}/\dot{F}^2$  with basis  $\{a_1, \dots, a_m\}$  where  $m \leq n$ , if  $\prod_{i=1}^m [a_i, x_i] = 1$  for some  $x_i \in \dot{F}$ , then there exist  $z_P \in N_F(a_P)$  for each  $P \subseteq \{1, 2, \dots, m\}$ , such that  $x_i = \prod_{P \ni i} z_P$ , for all  $i = 1, 2, \dots, m$ .*

*Proof.* This follows from the definition, using  $M = F(\sqrt{G})$ , by rearranging the terms of  $\varphi_M((z_P)) = \prod_P a_P \otimes z_P$ . Compare [30, (1.1)]. ■

1.5. LEMMA. *Every field  $F$  satisfies properties  $P_1(1)$  and  $P_1(2)$ .*

*Proof.* (See [30, (1.3)].) For  $P_1(2)$ , let  $M = F(\sqrt{a_1}, \sqrt{a_2})$ . Suppose  $[a_1, x_1][a_2, x_2] = 1$ . By (1.3) we must find  $z_1 \in N_F(a_1)$ ,  $z_2 \in N_F(a_2)$  and  $z \in N_F(a_1 a_2)$  with  $x_1 = z_1 z$  and  $x_2 = z_2 z$ . It suffices to find  $z \in \dot{F}$  such that

$$[a_1, x_1] = [a_1, z] = [a_2, z] = [a_2, x_2].$$

For, with such  $z$ , then  $z_1 = x_1 z$  and  $z_2 = x_2 z$  will do the job.

The existence of  $z$  is the well-known "common slot lemma." For proof, see [2, (6.3); 19, p. 69, Exercise 12] or more generally [1, (1.7)]. ■

At this point we can verify the property  $P_1$  for certain easy types of fields.

1.6. PROPOSITION. (1) *Suppose  $|\dot{F}/\dot{F}^2| = 2^d < \infty$ . If  $F$  satisfies  $P_1(d)$ , then  $F$  satisfies  $P_1$ .*

(2) Let  $m(F)$  be the number of isomorphism classes of quaternion algebras over  $F$ . Suppose  $m(F) = m < \infty$ . If  $F$  satisfies  $P_1(m-1)$ , then  $F$  satisfies  $P_1$ .

*Proof.* (1) Suppose  $M = F(\sqrt{G})$  where  $|G| = 2^n$ . Then  $n \leq d$  and hence  $P_1(n)$  holds.

(2) Suppose  $F$  satisfies  $P_1(n)$  for some  $n \geq m-1$ . To verify  $P_1(n+1)$  consider  $M = F(\sqrt{G})$  where  $G$  has basis  $\{a_1, \dots, a_{n+1}\}$ . Suppose  $\prod_{i=1}^{n+1} [a_i, x_i] = 1$ , for some  $x_i \in \dot{F}$ . If  $\prod_{i \in J} [a_i, x_i] = 1$  for some proper subset  $J \subseteq \{1, 2, \dots, n+1\}$ , then  $P_1(n)$  applied to the two complementary subproducts yields  $\prod_{i=1}^{n+1} a_i \otimes x_i \in \text{im } \phi_M$ . Hence, to prove  $P_1(n+1)$  we need only consider those cases where the algebras  $[a_i, x_i]$  satisfy no relations shorter than the given one. But then  $m \geq n+2$ , contrary to hypothesis, so these cases do not occur. ■

1.7. COROLLARY. If  $|\dot{F}/\dot{F}^2| \leq 4$  or if  $m(F) \leq 2$ , then  $F$  satisfies  $P_1$ . For instance, if  $F$  is a finite field or a local field, then  $F$  satisfies  $P_1$ .

With a little more work one can extend this result to fields  $F$  with  $m(F) \leq 4$ . No examples are known of a field  $F$  with  $|\dot{F}/\dot{F}^2| < \infty$  or with  $m(F) < \infty$  for which property  $P_1$  fails.

Suppose  $K$  is an intermediate field,  $F \subseteq K \subseteq M$ , in the multiquadratic extension  $M/F$ . We will derive an exact sequence relating the groups  $N_1(K/F)$  and  $N_1(M/F)$ .

If  $M = F(\sqrt{G})$  as above, then by Kummer theory we know that  $K = F(\sqrt{H})$  for some subgroup  $H \subseteq G$ . Also  $M/K$  is multiquadratic; in fact,  $M = K(\sqrt{G/H})$ , where we use the natural embedding of  $G/H$  in  $\dot{K}/\dot{K}^2$ . There is a natural map  $f$  from the complex  $\mathcal{B}_{K/F}$  to  $\mathcal{B}_{M/F}$ , giving an exact sequence of complexes:

$$1 \rightarrow \mathcal{B}_{K/F} \xrightarrow{f} \mathcal{B}_{M/F} \rightarrow \text{coker } f \rightarrow 1.$$

Written out, this sequence is a commutative diagram with exact rows, as in [30, Sect. 2].

1.8. Diagram.

$$\begin{array}{ccccccc} 1 \longrightarrow & \bigoplus_{h \in H} N_F(h) & \xrightarrow{f_1} & \bigoplus_{g \in G} N_F(g) & \longrightarrow & \bigoplus_{g \in G-H} N_F(g) & \longrightarrow 1 \\ & \downarrow \phi_K & & \downarrow \phi_M & & \downarrow \bar{\phi} & \\ 1 \longrightarrow & H \otimes (\dot{F}/\dot{F}^2) & \xrightarrow{f_2} & G \otimes (\dot{F}/\dot{F}^2) & \xrightarrow{\pi \otimes 1} & (G/H) \otimes (\dot{F}/\dot{F}^2) & \longrightarrow 1 \\ & \downarrow \phi_K & & \downarrow \phi_M & & \downarrow \bar{\phi} & \\ 1 \longrightarrow & \text{Br}_2(K/F) & \xrightarrow{f_3} & \text{Br}_2(M/F) & \longrightarrow & \text{coker } f_3 & \longrightarrow 1 \end{array}$$

The maps  $\bar{\varphi}$ ,  $\bar{\psi}$  in the last column are defined by the requirement that the diagram commute. The map  $\pi$  is the usual projection  $G \rightarrow G/H$ .

Let us give a name to the homology group of the last column of (1.8):

$$N_1(M/K/F) = \frac{\ker \bar{\psi}}{\text{im } \bar{\varphi}}.$$

1.9. LEMMA. *There is an exact sequence*

$$\ker \bar{\varphi} \rightarrow N_1(K/F) \xrightarrow{j} N_1(M/F) \xrightarrow{j} N_1(M/K/F) \xrightarrow{\Delta} N_2(K/F) \rightarrow N_2(M/F).$$

*Proof.* This is part of the long exact sequence of homology associated to diagram (1.8). Compare [30, (2.2)]. ■

1.10. COROLLARY. *If  $N_1(K/F) = N_2(K/F) = 1$ , then  $N_1(M/F) \cong N_1(M/K/F)$ . This always occurs if  $[K:F] \leq 4$ .*

*Proof.* The first statement is clear from (1.9). For the second we note that  $P_1(2)$  always holds, by (1.5), and  $P_2(2)$  always holds [11, (3.12); 29, (2.13); 30, (2.8)]. ■

We can compute the group  $N_1(M/K/F)$  when  $[M:K] = 2$ . In this case,  $H \subseteq G$  has index 2, so that  $G = H \cup aH$  for some  $a \in \dot{F}$ . Then  $K = F(\sqrt{H})$  and  $M = K(\sqrt{a})$ . We can identify  $G/H$  with  $\{1, a\} \subseteq \dot{F}/\dot{F}^2$ , so that every element of  $(G/H) \otimes (\dot{F}/\dot{F}^2)$  is of the form  $a \otimes x$ , for some  $x \in \dot{F}/\dot{F}^2$ .

1.11. LEMMA. *Let  $K = F(\sqrt{H})$  and  $M = K(\sqrt{a})$  as above. Suppose  $x \in \dot{F}$ .*

- (1)  $a \otimes x \in \ker \bar{\psi}$  iff  $[a, x]_K = 1$ , iff  $x \in \dot{F} \cap N_K(a)$ .
- (2) *The following are equivalent:*
  - (i)  $a \otimes x \in \text{im } \bar{\varphi}$ ,
  - (ii)  $x \in \prod_{h \in H} N_F(ah)$ ,
  - (iii)  $[a, x] = \prod_{h \in H} A_h$ , where  $A_h = [a, r_h] = [h, s_h]$  for some  $r_h, s_h \in \dot{F}$ .

(3) *The class of  $a \otimes x$  in  $N_1(M/K/F)$  lies in  $\text{im}(j: N_1(M/F) \rightarrow N_1(M/K/F))$  iff  $[a, x] \in \text{Dec}(K/F)$ .*

*Proof.* The product symbol  $\prod$  here denotes a product of elements or subgroups in a group, not the formal direct product. We prove (2), (3), leaving (1) for the reader.

(2) Suppose (i) holds, so that  $a \otimes x = \bar{\varphi}((z_h))$  for some  $(z_h)_{h \in H} \in \bigoplus_h N_F(ah)$ . Then  $a \otimes x = \prod_{h \in H} a \otimes z_h$  since  $\pi(ah) = a$ . Hence  $w = x \prod_h z_h$  lies in  $N_F(a)$ . Replacing  $z_1$  by  $wz_1$  yields  $x = \prod_h z_h$  as required for (ii).

Clearly (ii) implies (iii) using  $r_h = s_h = z_h$ . Assume (iii) holds. The "common slot lemma" noted in (1.5) implies the existence of  $z_h \in \dot{F}$  such that  $A_h = [a, z_h] = [h, z_h]$ , for each  $h \in H$ . Then  $z_h \in N_F(ah)$ , and we conclude  $a \otimes x = \bar{\varphi}((z_h))$ , proving (i).

(3) By the exactness of (1.9),  $a \otimes x \in \text{im } j$  iff  $\delta(a \otimes x) = 1$ , iff  $[a, x] \in \text{im } \psi_K = \text{Dec}(K/F)$ . ■

The following expression for  $N_1(M/K/F)$  is an immediate consequence:

1.12. COROLLARY. Suppose  $K = F(\sqrt{H})$  and  $M = K(\sqrt{a})$  as above. Then  $N_1(M/K/F) \cong (\dot{F} \cap N_K(a)) / \prod_{h \in H} N_F(ah)$ .

If  $a, b, c \in \dot{F}$ , define  $N_1(a, b, c) = N_1(M/F)$ , where  $M = F(\sqrt{a}, \sqrt{b}, \sqrt{c})$ .

1.13. COROLLARY. If  $a, b, c \in \dot{F}$ , let  $K = F(\sqrt{b}, \sqrt{c})$ . Then

$$N_1(a, b, c) \cong \frac{\dot{F} \cap N_K(a)}{N_F(a) N_F(ab) N_F(ac) N_F(abc)}.$$

*Proof.* Let  $H = \{1, b, c, bc\}$ . If  $a \in H$  the result is trivially true. If  $a \notin H$ , apply (1.10) and (1.12). ■

The next well-known lemma facilitates further calculations with these norm groups.

1.14. LEMMA. If  $E = F(\sqrt{c})$  and  $a \in \dot{F}$ , then  $\dot{F} \cap N_E(a) = N_F(a) N_F(ac)$ .

*Proof.* See [3, (3.5); 2, (4.1); 29, (1.10)]. ■

1.15. COROLLARY. If  $a, b, c \in \dot{F}$  and  $E = F(\sqrt{c})$ , then

$$N_1(a, b, c) \cong \frac{\dot{F} \cap N_E(a) N_E(b)}{(\dot{F} \cap N_E(a))(\dot{F} \cap N_E(b))}.$$

*Proof.* From (1.13) and (1.14) we find  $N_1(a, b, c) \cong [\dot{F} \cap N_E(a) N_E(ab)] / (\dot{F} \cap N_E(a))(\dot{F} \cap N_E(ab))$ . Since  $N_1(a, b, c)$  depends only on extension  $M/F$  where  $M = F(\sqrt{a}, \sqrt{b}, \sqrt{c})$ , we can replace  $b$  by  $ab$  to obtain the result. ■

Some similar formulas for the group  $N_1(a, b, c)$  appear in [29, (3.15)]. For later use we will isolate two further properties,  $R(n)$  and  $Q(n)$ . We investigate  $R(n)$  first.



1.16. DEFINITION.  $F$  has *Property  $R(n)$*  if, whenever  $K/F$  is multi-quadratic with  $[K:F] \leq 2^n$ , then every quaternion algebra in  $\text{Br}_2(K/F)$  lies in  $\text{Dec}(K/F)$ .

Since property  $P_2(n)$  states that  $\text{Br}_2(K/F) = \text{Dec}(K/F)$  for every such extension, this property  $R(n)$  is the restriction of  $P_2(n)$  to quaternion algebras. Since  $P_2(1)$  and  $P_2(2)$  always hold, by [30, (2.8)] or [11, (3.12)], then  $R(1)$  and  $R(2)$  always hold.

Generally,  $F$  satisfies  $P_2(n)$  iff  $F$  satisfies  $R(n)$  and for every multi-quadratic  $K/F$  with  $[K:F] \leq 2^n$ ,  $\text{Br}_2(K/F)$  is generated by quaternion algebras. When  $n = 3$  then  $\text{Br}_2(K/F)$  is always generated by quaternions. In fact [30, (3.2)], every element of  $N_2(K/F)$  can be represented by one quaternion algebra in  $\text{Br}_2(K/F)$ . Consequently, the properties  $R(3)$  and  $P_2(3)$  are equivalent. The examples in [11, Sect. 5] then show that  $R(3)$  does not always hold.

The property  $R(n)$  can be formulated in terms of the long exact sequence of (1.9).

1.17. LEMMA.  $F$  satisfies  $R(n)$  iff the map  $j: N_1(M/F) \rightarrow N_1(M/K/F)$  is surjective, whenever  $F \subseteq K \subseteq M$  is multi-quadratic with  $[K:F] \leq 2^n$  and  $[M:K] = 2$ .

*Proof.* Suppose  $K = F(\sqrt{H})$  and  $M = K(\sqrt{a})$  as usual. An element of  $N_1(M/K/F)$  is represented by some  $a \otimes x \in \ker \bar{\psi}$ . That is,  $[a, x] \in \text{Br}_2(K/F)$  by (1.11)(1). The equivalence with  $R(n)$  is then seen from (1.11)(3). ■

1.18. PROPOSITION.  $F$  satisfies  $P_1(n+1)$  and  $R(n)$  iff  $N_1(M/K/F) = 1$  for every multi-quadratic  $F \subseteq K \subseteq M$  with  $[K:F] \leq 2^n$  and  $[M:K] = 2$ .

*Proof.* If  $F$  has  $P_1(n+1)$  and  $R(n)$ , then  $N_1(M/K/F) = 1$  by (1.9) and (1.17). For the converse, we first note that  $R(n)$  is trivially true by (1.17), and then prove  $P_1(m)$  inductively for  $m \leq n+1$ , using (1.9). ■

The property  $Q(n)$  is designed to be the inductive step from  $P_1(n)$  to  $P_1(n+1)$ . It states that every quaternion algebra in  $\text{Dec}(K/F)$  has an even nicer decomposition.

1.19. DEFINITION.  $F$  has *Property  $Q(n)$*  if, whenever  $K = F(\sqrt{H})$  is a multi-quadratic extension with  $[K:F] = 2^n$ , and  $a, x \in \dot{F}$ , then  $[a, x] \in \text{Dec}(K/F)$  implies that  $x \in \prod_{h \in H} N_F(ah)$ .

In view of (1.11)(2) this implication can be restated as follows: if  $[a, x] \in \text{Dec}(K/F)$ , then  $[a, x] = \prod_{h \in H} A_h$ , where  $A_h = [a, r_h] = [h, s_h]$  for some  $r_h, s_h \in \dot{F}$ .

1.20. LEMMA. *F satisfies  $Q(n)$  iff whenever  $F \subseteq K \subseteq M$  is multiquadratic with  $[K:F] = 2^n$  and  $[M:K] = 2$ , the map  $j: N_1(M/F) \rightarrow N_1(M/K/F)$  is trivial.*

*Proof.* “if.” Suppose  $K = F(\sqrt{H})$  is given with  $[K:F] = 2^n$  and  $[a, x] \in \text{Dec}(K/F)$ . If  $a \in H$  the property is trivial, so assume  $a \notin H$  and let  $M = K(\sqrt{a})$ . Suppose  $[a, x] = \prod_{h \in H} [h, y_h]$  for some  $y_h \in \dot{F}$ . Let  $\alpha = a \otimes x + \sum_{h \in H} h \otimes y_h$ , so that  $\alpha \in \ker \psi$ . The triviality of  $j$  says that  $a \otimes x = (\pi \otimes 1)(\alpha)$  lies in  $\text{im } \bar{\varphi}$ . The condition for  $Q(n)$  now follows from (1.11)(2).

“only if.” Suppose  $K = F(\sqrt{H})$  and  $M = K(\sqrt{a})$  as usual, with  $G = H \cup aH$ . Any  $\alpha \in G \otimes (\dot{F}/\dot{F}^2)$  can be written as  $\alpha = a \otimes x + \sum_{h \in H} h \otimes y_h$ , for some  $x, y_h \in \dot{F}$ . If  $\alpha \in \ker \psi$ , then  $[a, x] = \prod [h, y_h] \in \text{Dec}(K/F)$ . Applying  $Q(n)$  and (1.11)(2), we have  $a \otimes x \in \text{im } \bar{\varphi}$ . Therefore,  $j$  is trivial on the class of  $\alpha$  in  $N_1(M/F)$ . ■

1.21. PROPOSITION. *F satisfies  $P_1(n+1)$  iff F satisfies  $P_1(n)$  and  $Q(n)$ . Hence, F satisfies  $P_1(n)$  iff F satisfies  $Q(m)$  for  $2 \leq m \leq n-1$ .*

*Proof.* The first statement follows from (1.9) and (1.20). The second is then clear, using (1.5). ■

## 2. CONNECTIONS WITH QUADRATIC FORM THEORY

The properties  $P_1(n)$  have some interesting connections with piecewise equivalence and common value properties of quadratic forms, and with representational abstract Witt rings. In this section we describe these ideas, using some results proved in [17, 28].

We begin by describing an equivalence relation for quadratic forms associated to a subgroup  $G \subseteq \dot{F}/\dot{F}^2$ . Here,  $G$  is often identified with its lift in  $\dot{F}$ . Let  $\alpha = \langle a_1, \dots, a_n \rangle$  and  $\beta = \langle b_1, \dots, b_n \rangle$  be two diagonal  $F$ -quadratic forms of the same dimension, following the notation of [19]. Define  $\alpha \sim_G \beta$  if either

- (i) for some  $j, k$ ,  $\langle a_j, a_k \rangle \cong \langle b_j, b_k \rangle$  and  $a_i = b_i$  for all  $i \neq j, k$ ; or
- (ii) for some  $j$ ,  $a_j b_j \in G$  and  $a_i = b_i$  for all  $i \neq j$ .

As in [28, (1.3)], the equivalence relation generated by  $\sim_G$  is called  $G$ -equivalence and denoted  $\approx_G$ .

In the case  $|G|=1$  (that is,  $G = \dot{F}^2$ ),  $\approx_G$  is the same as  $\cong$ , the isometry relation. This is a theorem of Witt [19, p. 21]. Therefore, for any  $G$ ,  $G$ -equivalence depends only on the isometry classes of the quadratic forms.

If  $K = F(\sqrt{G})$ , then  $\alpha \approx_G \beta$  implies  $\alpha_K \cong \beta_K$ . (Here  $q_K = q \otimes K$ .) We will

define an intermediate relation in terms of the Witt ring ideal  $\mathfrak{U}(G)$ . Let  $WF$  be the Witt ring of anisotropic quadratic forms over  $F$ , as in [19].

**2.1. DEFINITION.** Let  $G \subseteq \dot{F}/\dot{F}^2$  be a subgroup. Define  $\mathfrak{U}(G)$  to be the ideal in  $WF$  generated by all the binary forms  $\langle\langle -g \rangle\rangle = \langle 1, -g \rangle$ , for  $g \in G$ . That is,

$$\mathfrak{U}(G) = \sum_{g \in G} \langle\langle -g \rangle\rangle WF.$$

Note that  $\mathfrak{U}(G) \subseteq W(K/F) = \ker(WF \rightarrow WK)$ , where  $K = F(\sqrt{G})$ . We say that  $K/F$  has the *kernel property* if  $\mathfrak{U}(G) = W(K/F)$ . Recall from [12] that  $F$  is called *1-amenable* if every multiquadratic  $K/F$  has this kernel property. In [11] these properties are investigated further, using the notation  $WD(K/F)$  rather than our  $\mathfrak{U}(G)$ .

If  $\alpha, \beta$  are  $n$ -dimensional  $F$ -forms, then  $\alpha \equiv \beta \pmod{\mathfrak{U}(G)}$  implies  $\alpha_K \cong \beta_K$ . Also it is easy to check that  $\alpha \approx_G \beta$  implies  $\alpha \equiv \beta \pmod{\mathfrak{U}(G)}$ . We are interested in the converses, especially for binary forms.

**2.2. DEFINITION.** (1)  $G$  satisfies *binary PE* (piecewise equivalence) if, whenever  $\alpha, \beta$  are binary  $F$ -forms, then  $\alpha \equiv \beta \pmod{\mathfrak{U}(G)}$  implies  $\alpha \approx_G \beta$ .

(2)  $G$  satisfies *binary APE* (amenable piecewise equivalence) if, whenever  $\alpha, \beta$  are binary  $F$ -forms and  $K = F(\sqrt{G})$ , then  $\alpha_K \cong \beta_K$  implies  $\alpha \approx_G \beta$ .

The stronger properties PE and APE are defined similarly, requiring only that the  $F$ -forms  $\alpha, \beta$  be of equal dimension, rather than being binary.

Clearly  $G$  has binary APE iff  $G$  has binary PE and  $K/F$  has the kernel property for 4-dimensional forms. If  $|G| \leq 2$ , then  $G$  does satisfy APE [13, (3.12); 31, Proposition 6].

**2.3. THEOREM.** Let  $G \subseteq \dot{F}/\dot{F}^2$  be a finite subgroup and let  $K = F(\sqrt{G})$ . Then,  $G$  has binary APE iff  $\dot{F} \cap N_K(a) = \prod_{g \in G} N_F(ag)$ , for every  $a \in \dot{F}$ . That is,  $G$  has binary APE iff  $N_1(K(\sqrt{a})/K/F) = 1$  for all  $a \in \dot{F}$ .

*Proof.* The first statement appears in [28, (2.3)]. For the second, apply (1.12). ■

**2.4. COROLLARY.** (1)  $F$  satisfies properties  $P_1(n+1)$  and  $R(n)$  iff every subgroup  $G \subseteq \dot{F}/\dot{F}^2$  with  $|G| \leq 2^n$  satisfies binary APE.

(2)  $F$  satisfies  $P_1(3)$  iff every subgroup  $G \subseteq \dot{F}/\dot{F}^2$  with  $|G| = 4$  has binary PE.

*Proof.* (1) Use (2.3) and (1.18).

(2) We know property  $R(2)$  always holds. Also, if  $|G| \leq 4$ , then  $K/F$  has the kernel property [12, (2.12)], so that APE and PE are equivalent in this case. The claim now follows from part (1). ■

**2.5. Remark.** Fields  $F$  are exhibited in Section 5 which do not satisfy  $P_1(3)$ . Hence there are groups  $G \subseteq \dot{F}/\dot{F}^2$  with  $|G| = 4$  not satisfying binary PE. These provide the first known examples where property PE fails.

The investigation of piecewise equivalence led to the study of certain common value properties for quadratic forms. If  $E/F$  is a field extension and  $q$  is an  $F$ -form, let  $D_E(q)$  be the set of values in  $\dot{E}$  represented by  $q_E = q \otimes E$ .

**2.6. DEFINITION** [28, (2.5)]. A field extension  $E/F$  satisfies *Property CV*( $m, n$ ) when, for any quadratic forms  $\alpha, \beta$  over  $F$  with  $\dim \alpha = m$  and  $\dim \beta = n$ , if  $(\alpha \perp -\beta)_E$  is isotropic, then  $D_E(\alpha) \cap D_E(\beta) \cap \dot{F} \neq \emptyset$ . The extension  $E/F$  satisfies *Property CV* if it satisfies  $CV(m, n)$  for all  $m, n$ .

Note that the hypothesis  $(\alpha \perp -\beta)_E$  isotropic is equivalent to  $D_E(\alpha) \cap D_E(\beta) \neq \emptyset$ . The properties  $CV(1, n)$  and  $CV(m, 1)$  are trivially true.

Using the method of [12, (2.9)] the property CV can be simplified:

**2.7. LEMMA.** Suppose  $[E:F] = l$ . Then  $E/F$  has CV iff  $E/F$  has  $CV(m, n)$  for all  $m, n \leq l$ .

It can be proved [28, (2.10)] that if  $E/F$  has CV, then  $E/F$  is excellent, in the sense of [12]. For some types of fields it is known that CV and excellence are equivalent (e.g., global fields [22]).

Let us consider property  $CV(2, 2)$  more closely. Let  $\alpha = \langle 1, -a \rangle$  and  $\beta = x \langle 1, -b \rangle$  over  $F$ , and let  $E/F$  be a field extension. A direct calculation shows:

- (1)  $(\alpha \perp -\beta)_E$  is isotropic iff  $x \in \dot{F} \cap N_E(a) N_E(b)$ .
- (2)  $D_E(\alpha) \cap D_E(\beta) \cap \dot{F} \neq \emptyset$  iff  $x \in (\dot{F} \cap N_E(a))(\dot{F} \cap N_E(b))$ .

**2.8. PROPOSITION.**  $F$  satisfies  $P_1(3)$  iff every quadratic extension  $E/F$  satisfies CV.

*Proof.* By (1.15), (2.7) and the remarks above, if  $E = F(\sqrt{c})$ , then  $E/F$  satisfies CV iff  $E/F$  satisfies  $CV(2, 2)$ , iff  $N_1(a, b, c) = 1$  for all  $a, b \in \dot{F}$ . ■

Therefore the fields described in Section 5 provide examples of quadratic extensions where  $CV(2, 2)$  fails. (See (5.4).) Since every quadratic extension is excellent, this shows that CV and excellence are not in general equivalent.

2.9. PROPOSITION. *The following statements are equivalent for a field  $F$ .*

- (1) *Every subgroup  $G \subseteq \dot{F}/\dot{F}^2$  has binary APE.*
- (2) *Every multiquadratic  $K/F$  satisfies CV(2, 2).*
- (3)  *$F$  satisfies properties  $P_1$  and  $R$ .*

*Proof.* The equivalence of (1) and (2) is proved in [28, (2.8)]. The equivalence of (1) and (3) follows from (2.4) and a straightforward reduction to the cases of finite subgroups and finite extensions. ■

It is interesting to note from (2.9) and (1.3) that assumptions about the behavior of binary  $F$ -forms under multiquadratic extensions are sufficient to imply the injectivity of the quaternion symbol  $g_F: k_2 F \rightarrow \text{Br}_2(F)$ .

2.10. Remark. There are several known classes of fields satisfying these properties  $P_1$  and  $R$ , including (i) local fields, (ii) global fields, (iii) fields with Hasse number  $\tilde{u} \leq 2$ , (iv) Pythagorean fields with finite chain length, (v) a direct limit of fields satisfying  $P_1$  and  $R$ . For proofs, or proof sketches, see (1.7), (3.7), (3.6), (3.11) and [11, (3.18), (3.19)].

Another interesting connection with property  $P_1(3)$  is given by representational abstract Witt rings, as first pointed out by Lam. For a field extension  $E/F$ , consider the ring

$$R = R(E/F) = \text{im}(i: WF \rightarrow WE),$$

where  $i$  is the map between Witt rings given by extending scalars. Then  $R$  can be viewed as an abstract Witt ring for the group  $G = \dot{F}/\dot{F}^2$ , in the sense of [17, 18].

A form for  $R$  is given by the quadratic form  $\alpha$  over  $F$ , where two forms  $\alpha, \beta$  are  $R$ -equivalent if  $\alpha_E \cong \beta_E$ . The set of  $R$ -represented values of  $\alpha$  is

$$D_R(\alpha) = \{x \in \dot{F} \mid \alpha_E \cong \langle x \rangle_E \perp \alpha'_E, \text{ for some } F\text{-form } \alpha'\}.$$

Similarly, a form  $\alpha$  is said to be  $R$ -isotropic if  $\alpha_E \cong \langle 1, -1 \rangle \perp \alpha''_E$ , for some  $F$ -form  $\alpha''$ .

In the special case  $E/F$  is excellent, these definitions are easier to handle. In fact, the definition of excellence [12] can be restated:  $E/F$  is excellent iff for every  $F$ -form  $\alpha$ , if  $\alpha_E$  is isotropic, then  $\alpha$  is  $R$ -isotropic. Equivalently, for every  $F$ -form  $\alpha$ ,  $D_R(\alpha) = \dot{F} \cap D_E(\alpha)$ .

The abstract Witt ring  $R = R(E/F)$  is said to be *representational* [17, (2.2)] if whenever  $\sigma, \tau$  are  $F$ -forms and  $x \in D_R(\sigma \perp \tau)$ , then there exists  $s \in D_R(\sigma)$  and  $t \in D_R(\tau)$  with  $x \in D_R(\langle s, t \rangle)$ . This property can be restated to look like a common value property:

2.11. LEMMA [17, (2.4)].  *$R$  is representational iff whenever  $\alpha, \beta$  are  $F$ -forms with  $\alpha \perp -\beta$   $R$ -isotropic, then  $D_R(\alpha) \cap D_R(\beta) \neq \emptyset$ .*

2.12. PROPOSITION (Lam).  *$E/F$  satisfies property CV iff  $E/F$  is excellent and  $R(E/F)$  is representational.*

*Proof.* CV implies excellence [28], and the rest follows by the remarks above. ■

2.13. COROLLARY.  *$F$  satisfies  $P_1(3)$  iff  $R(E/F)$  is representational, for every quadratic extension  $E/F$ .*

*Proof.* Use (2.8), (2.12) and the fact that every quadratic extension is excellent. ■

If  $E = F(\sqrt{c})$  then  $R(E/F) \cong WF/\langle\langle -c \rangle\rangle WF$ , by [19, p. 200]. Thus, a field  $F$  where  $P_1(3)$  fails provides an example of a representational Witt ring  $R_0 = WF$  having a quotient  $R = R_0/\langle\langle -c \rangle\rangle R_0$  which is not representational. This answers a question of Marshall.

For the convenience of the reader we will list the characterizations of the property  $P_1(3)$  scattered throughout this section.

2.14. COROLLARY. *The following are equivalent for a field  $F$ .*

- (1)  *$P_1(3)$  holds for  $F$ .*
- (2)  *$N_1(a, b, c) = 1$  for every  $a, b, c \in \dot{F}$ .*
- (3) *If  $G \subseteq \dot{F}/\dot{F}^2$  is a group with  $|G| = 4$ , then  $G$  has binary PE.*
- (4) *Every quadratic extension  $E/F$  satisfies CV.*
- (5)  *$WF/\langle\langle -a \rangle\rangle WF$  is representational, for every  $a \in \dot{F}$ .*

*Proof.* See (1.13) for  $N_1(a, b, c)$  and (2.4), (2.8) and (2.13) for the other equivalences. ■

### 3. FIELDS WITH PROPERTY $P_1$

The main purpose of this section is to present proofs of property  $P_1$  for fields  $F$  with Hasse number  $\tilde{u}(F) \leq 2$ , and for global fields. We begin by introducing the complex  $\mathcal{E}_{M/F}$ , proving it is exact and considering the natural homomorphism of complexes  $\mathcal{B}_{M/F} \rightarrow \mathcal{E}_{M/F}$  induced by the signature maps. These ideas illuminate the local-global methods used to prove  $P_1$  in the case  $\tilde{u}(F) \leq 2$ . A somewhat different local-global argument is used in the case of global fields. Some further examples of fields having  $P_1$  are mentioned without proof at the end of the section.

For a field  $F$ , let  $X_F$  be the topological space of orderings of  $F$ . Recall (see, e.g., [20], [21] or [27]) that the set  $X_F$  is given the topology where the subsets  $H(a)$  for  $a \in F$  form a subbase. Here

$$H(a) = \{\alpha \in X_F \mid a \text{ is positive at } \alpha\}.$$

Each  $H(a)$  is a clopen (closed and open) set. We write  $H(a_1, \dots, a_n)$  for  $H(a_1) \cap \dots \cap H(a_n)$ . In the notation of [11, Sect. 6],  $H(a) = \text{supp}\langle\langle a \rangle\rangle$ .

Define  $\bar{C}(F) = \mathcal{C}(X_F, \mathbb{Z}/2\mathbb{Z})$ , the ring of continuous functions from  $X_F$  to  $\mathbb{Z}/2\mathbb{Z}$ . (As usual,  $\mathbb{Z}/2\mathbb{Z}$  is given the discrete topology.) If  $S \subseteq X_F$  is clopen, then the characteristic function  $\chi_S$  lies in  $\bar{C}(F)$ . In fact, every  $f \in \bar{C}(F)$  equals  $\chi_S$  for some clopen set  $S$ , so that  $\bar{C}(F)$  can be identified with the Boolean algebra of all clopen subsets of  $X_F$ . This view point will prove useful later.

If  $M/F$  is a field extension, let  $\varepsilon = \varepsilon_{M/F}: X_M \rightarrow X_F$  be the restriction map. This induces a ring homomorphism  $\rho = \rho_{M/F}: \bar{C}(F) \rightarrow \bar{C}(M)$  defined by  $\rho(f) = f \circ \varepsilon$ .

For quadratic extensions  $L/F$  we need the additive homomorphism  $\bar{s}_\# = \bar{s}_\#^{L/F}: \bar{C}(L) \rightarrow \bar{C}(F)$  induced by the transfer map  $s_\#$  defined in [11, (6.6)]. Let us state this definition explicitly. If  $\alpha \in X_F$ , then either  $\alpha$  does not extend to  $L$ , or  $\alpha$  extends to exactly two orderings  $\beta, \bar{\beta} \in X_L$ . Here the “bar” denotes the action of the nontrivial element of the Galois group on  $X_L$ . Then if  $f \in \bar{C}(L)$  and  $\alpha = \beta|_F$  where  $\beta \in X_L$ , then

$$\bar{s}_\#(f)(\alpha) = f(\beta) + f(\bar{\beta}) \quad \text{in } \mathbb{Z}/2\mathbb{Z}.$$

If  $\alpha \in X_F$  does not extend to  $L$ , define  $\bar{s}_\#(f)(\alpha) = 0$ . Note that this transfer map  $\bar{s}_\#$  depends only on the quadratic extension  $L/F$ , and not on any choice of a linear map  $s: L \rightarrow F$ .

Suppose  $M = F(\sqrt{G})$  is a multiquadratic extension of  $F$  and consider the following sequence:

$$\mathcal{C}_{M/F}: \bigoplus_{L \in Q(M/F)} \bar{C}(L) \xrightarrow{\psi'} G \otimes \bar{C}(F) \xrightarrow{\psi'} \bar{C}(F) \xrightarrow{\rho} \bar{C}(M). \quad (3.1)$$

If  $a \otimes f \in G \otimes \bar{C}(F)$ , then we define  $\psi'(a \otimes f) = \chi_{H(-a)} \cdot f$ , the product in the ring  $\bar{C}(F)$ . If  $(f_L) \in \bigoplus \bar{C}(L)$ , we define

$$\psi'((f_L)) = \sum_{L \in Q(M/L)} \kappa(L) \otimes \bar{s}_\#^{L/F}(f_L).$$

Generally if  $L = F(\sqrt{a})$ , then  $\alpha \in X_F$  extends to an ordering of  $L$  iff  $\alpha$  is positive at  $a$ . In particular, if  $a \in G$ , then  $H(-a)$  is disjoint from  $X_{M/F} = \text{im } \varepsilon$ , and hence  $\rho \circ \psi' = 0$ . Also if  $\alpha \in H(-a)$ , then  $\alpha$  does not extend to  $L$ , and we have  $\bar{s}_\#^{L/F}(f)(\alpha) = 0$  for every  $f \in \bar{C}(L)$ . It follows that  $\psi' \circ \varphi' = 0$ . Therefore  $\mathcal{C}_{M/F}$  is complex.

3.2. *Remark.* A complex  $\mathcal{C}_{M/F}$  is defined in [11, Sect. 6] using the rings  $C(F) = \mathcal{C}(X_F, \mathbb{Z})$ , while we are using the complex  $\mathcal{C}_{M/F}$  formed from rings  $\bar{C}(F) \cong C(F)/2C(F)$ . We use this “mod 2” version of the complex because there seems to be no obvious way to extend  $\mathcal{C}_{M/F}$  to include the tensor product term. In particular note that  $G \otimes C(F)$  is 2-torsion while  $C(F)$  is torsion free.

3.3. **THEOREM.** *If  $M/F$  is a multiquadratic extension, then the sequence  $\mathcal{C}_{M/F}$  is exact.*

In order to prove this exactness it is convenient to change our viewpoint. If  $S \subseteq X_F$  is a clopen set, we will use the same letter  $S$  to denote the corresponding element of  $\bar{C}(F)$  (formerly written  $\chi_S$ ). The product of the clopen sets  $S, T$  is then  $S \cap T$ , and the sum  $S + T$  is the symmetric difference  $(S - T) \cup (T - S)$ . We will calculate the maps  $\phi'$  and  $\psi'$  using this Boolean notation.

This Boolean notation provides an easy proof of the exactness of  $\mathcal{C}_{M/F}$  at  $\bar{C}(F)$ . If  $g \otimes S \in G \otimes \bar{C}(F)$ , then  $\psi'(g \otimes S) = H(-g) \cap S$ . Also it is easily checked that for  $S \in \bar{C}(F)$  we have  $\rho(S) = \varepsilon^{-1}(S)$  in  $\bar{C}(M)$ . Then  $S \in \ker \rho$  iff  $\varepsilon^{-1}(S)$  is empty, iff  $S \subseteq \bigcup_{g \in G} H(-g)$ . For any such  $S$  we can express  $S = \bigcup_{g \in G} S_g$  where these  $S_g$  are disjoint clopen sets in  $X_F$  with  $S_g \subseteq H(-g)$ , for each  $g \in G$ . Using the Boolean sum, we have  $S = \sum S_g = \psi'(\sum g \otimes S_g) \in \text{im } \psi'$ . This proves the exactness of  $\mathcal{C}_{M/F}$  at  $\bar{C}(F)$ . This exactness is also a special case of [11, (7.5)]. The remainder of the proof of (3.3) is given after the next lemma.

3.4. **LEMMA.** *Let  $L = F(\sqrt{a})$  and  $S \in \bar{C}(F)$ . Then  $S \in \text{im } \bar{s}_\#^{L/L}$  iff  $S \subseteq H(a)$ .*

*Proof.* We first show that if  $T \in \bar{C}(L)$ , then  $\bar{s}_\#(T) = \varepsilon(T + \bar{T})$ . From the definition of  $\bar{s}_\#$  translated into Boolean notation, we have for  $T \in \bar{C}(L)$  and  $\alpha \in X_F$  that  $\alpha \in \bar{s}_\#(T)$  iff  $\alpha$  extends to orderings  $\beta, \bar{\beta}$  in  $X_L$  and exactly one of  $\beta, \bar{\beta}$  lies in  $T$ . Hence,  $\bar{s}_\#(T) = \varepsilon(T + \bar{T})$ , by the definition of Boolean sum.

Now if  $S \in \text{im } \bar{s}_\#$  then every  $\alpha \in S$  does extend to  $L$ , so  $S \subseteq H(a)$ . Conversely, suppose  $S \in \bar{C}(F)$  and  $S \subseteq H(a)$ . Given the choice of the square root  $\sqrt{a} \in L$ , we define  $T = H_L(\sqrt{a}) \cap \varepsilon^{-1}(S) = \{\beta \in X_L : \beta|_F \in S \text{ and } \sqrt{a} \text{ is positive at } \beta\}$ . Then  $T \cap \bar{T} = \emptyset$  and we have  $\bar{s}_\#(T) = S$ . ■

We will now complete the proof of Theorem 3.3, by proving exactness at  $G \otimes \bar{C}(F)$ . For an element  $\xi \in G \otimes \bar{C}(F)$ , we will reduce  $\xi$  to a “normal form” modulo  $\text{im } \phi'$ .

By (3.4), the group  $\text{im } \phi'$  is generated by all elements  $a \otimes S \in G \otimes \bar{C}(F)$  where  $S \subseteq H(a)$ . Suppose  $\{g_1, \dots, g_n\}$  is a basis of  $G$ . Then  $\xi \in G \otimes \bar{C}(F)$



can be written as  $\xi = \sum_{i=1}^n g_i \otimes S_i$ . The elements  $g_i \otimes (S_i \cap H(g_i))$  lie in  $\text{im } \varphi'$ , so we may alter  $\xi$  by these elements to assume  $S_i \subseteq H(-g_i)$  for each  $i$ .

Next, consider  $g_1 \otimes S_1 + g_2 \otimes S_2$  and define  $T_1 = S_1 - S_2 = S_1 + (S_1 \cap S_2)$  and  $T_2 = S_2 - S_1 = S_2 + (S_1 \cap S_2)$ . Then  $T_1 \cap T_2 = \emptyset$  and  $g_1 \otimes S_1 + g_2 \otimes S_2 = g_1 g_2 \otimes (S_1 \cap S_2) + g_1 \otimes T_1 + g_2 \otimes T_2$ . Since  $S_1 \cap S_2 \subseteq H(-g_1) \cap H(-g_2) \subseteq H(g_1 g_2)$ , we have  $g_1 g_2 \otimes (S_1 \cap S_2) \in \text{im } \varphi'$ . Altering  $\xi$  by this element, we may assume  $S_1 \cap S_2 = \emptyset$ . Continuing this process for the other pairs of indices, we find that  $\xi \equiv \sum_{i=1}^n g_i \otimes U_i \pmod{\text{im } \varphi'}$ , where the  $U_i$  are pairwise disjoint clopen sets and  $U_i \subseteq H(-g_i)$ , for all  $i$ .

Now suppose  $\xi \in \ker \psi'$ . Then  $0 = \psi'(\xi) \equiv \sum H(-g_i) \cap U_i = \sum U_i$ , so each  $U_i = \emptyset$  since they are disjoint. Therefore  $\xi \in \text{im } \varphi'$ . This completes the proof of (3.3). ■

Next we will describe the homomorphism of complexes  $\mathcal{B}_{M/F} \rightarrow \overline{\mathcal{C}}_{M/F}$ , given by the signature maps. The original signature map is a ring homomorphism  $\text{sgn}: WF \rightarrow C(F)$ . Using the isomorphism  $\dot{I}^1 F \cong \dot{F}/\dot{F}^2$  we have an induced map  $\text{sgn}: \dot{F}/\dot{F}^2 \rightarrow \overline{C}(F)$ , sending  $a \in \dot{F}/\dot{F}^2$  to  $\frac{1}{2} \text{sgn} \langle -a \rangle \in \overline{C}(F)$ . In the Boolean notation, this map sends  $a$  to  $H(-a) \in \overline{C}(F)$ .

There is also a natural map  $\text{sgn}: \text{Br}_2(F) \rightarrow \overline{C}(F)$  described as follows. If  $\alpha \in X_F$ , let  $F_\alpha$  denote the real algebraic closure of  $(F, \alpha)$ , and recall that  $\text{Br}_2(F_\alpha) \cong \mathbb{Z}/2\mathbb{Z}$ . Then for  $A \in \text{Br}_2(F)$ , define  $(\text{sgn } A)(\alpha) = A_\alpha$ , the image of  $A$  in  $\text{Br}_2(F_\alpha)$ . In the Boolean notation,  $\text{sgn } A = \{\alpha \in X_F \mid A \otimes F_\alpha \text{ is not split}\}$ . In particular if  $A = [a, b]$  for  $a, b \in \dot{F}$ , then  $\text{sgn}[a, b] = H(-a, -b)$ .

**3.5. PROPOSITION.** *For any multiquadratic extension  $M = F(\sqrt{G})$ , the following diagram is commutative.*

$$\begin{array}{ccccccc}
 \bigoplus_{L \in Q(M/F)} (\dot{L}/\dot{L}^2) & \longrightarrow & G \otimes (\dot{F}/\dot{F}^2) & \xrightarrow{\psi} & \text{Br}_2(F) & \xrightarrow{r} & \text{Br}_2(M) \\
 \downarrow \oplus \text{sgn} & & \downarrow 1 \otimes \text{sgn} & & \downarrow \text{sgn} & & \downarrow \text{sgn} \\
 \bigoplus_{L \in Q(M/F)} \overline{C}(L) & \xrightarrow{\omega'} & G \otimes \overline{C}(F) & \xrightarrow{\omega'} & \overline{C}(F) & \xrightarrow{p} & \overline{C}(M).
 \end{array}$$

*Proof.* The commutativity of the leftmost square follows from [11, (6.7)] after reducing mod 2. We can prove it directly by showing that, for any  $c \in \dot{L}$ , we have  $\tilde{s}_\#^{L/F}(H_L(-c)) = H_F(-N_{L/F}(c))$ . This equality follows by the techniques used to prove (3.4). The commutativity of the middle square is easy to check. For the rightmost square, suppose  $A \in \text{Br}_2(F)$ . If  $\alpha \in X_F$  extends to  $\gamma \in X_M$ , then  $A$  splits over  $F_\alpha$  iff it splits over  $M_\gamma$ . (In fact,  $F_\alpha \cong M_\gamma$  over  $F$ .) ■

As an application of these results on  $\mathcal{C}_{M/F}$  we consider fields  $F$  with  $\tilde{u}(F) \leq 2$ . Recall [5] that the *Hasse number*  $\tilde{u}(F)$  is  $\sup\{\dim q\}$ , where  $q$  ranges over all anisotropic indefinite quadratic forms over  $F$ . Several equivalent versions of the property  $\tilde{u}(F) \leq 2$  are given in [5, (4.8); 10 Theorem F]. It follows that  $\tilde{u}(F) \leq 2$  iff  $F$  is a SAP field and  $I^2F$  is torsion free. Since  $F$  has SAP, the map  $\text{sgn}: (\bar{F}/\bar{F}^2) \rightarrow \bar{C}(F)$  is surjective. (That is,  $F$  is reduced 1-stable [11, (8.3)].)

3.6. THEOREM. *If  $\tilde{u}(F) \leq 2$ , then  $F$  satisfies property  $P_1$ .*

*Proof.* Suppose  $M = F(\sqrt{G})$  is multiquadratic and  $\bar{\delta} \in N_1(M/F)$ . Then a representative  $\delta \in G \otimes (\bar{F}/\bar{F}^2)$  lies in  $\ker \psi$ . Using [15, (3.3)] we have  $\tilde{u}(L) \leq 2$  for every quadratic extension  $L/F$ , so the map  $\oplus \text{sgn}$  in (3.5) is surjective. By the exactness of  $\mathcal{C}_{M/F}$  and the surjectivity of  $\oplus \text{sgn}$ , a diagram chase shows that we may alter  $\delta$  by an element of  $\text{im } \varphi$  to assume  $(1 \otimes \text{sgn})(\delta) = 0$ .

Let  $\{g_1, \dots, g_n\}$  be a basis of  $G$  and express  $\delta = \sum g_i \otimes x_i$ . Then in the Boolean notation,  $0 = \sum g_i \otimes H(-x_i)$  in  $G \otimes \bar{C}(F)$ . The independence of the  $g_i$  then implies that  $H(-x_i) = \emptyset$ , so that  $x_i$  is totally positive in  $F$ , for each  $i$ . Then the form  $\langle -g_i, -x_i \rangle$  is totally indefinite, hence isotropic, since  $\tilde{u}(F) \leq 2$ . Therefore  $x_i \in N_F(g_i)$  for each  $i$ , and hence  $\delta \in \text{im } \varphi$ . We conclude that  $\bar{\delta} = 0$  in  $N_1(M/F)$ . ■

For example, suppose  $F/R$  is an extension of transcendence degree  $\leq 1$ , where  $R$  is a real closed field. Then  $\tilde{u}(F) \leq 2$  by [10, Theorem I], so such a field does have property  $P_1$ .

Our next theorem deals with global fields. The property  $P_1$  is proved in this case by a local-global method without explicit mention of an associated "local" complex.

3.7. THEOREM. *If  $F$  is a global field, then  $F$  satisfies  $P_1$ .*

For a global field  $F$ , let  $\mathbb{P}_F$  be the set of all prime spots of  $F$  (finite or infinite). If  $\mathfrak{p} \in \mathbb{P}_F$ , then  $F_{\mathfrak{p}}$  denotes the completion of  $F$  at  $\mathfrak{p}$ . Recall that  $\text{Br}_2(F_{\mathfrak{p}}) \cong \mathbb{Z}/2\mathbb{Z}$  for all  $\mathfrak{p} \in \mathbb{P}_F$  except the infinite complex spots. Using multiplicative notation now, we write  $\text{Br}_2(F_{\mathfrak{p}}) = \{1, -1\}$ . If  $A \in \text{Br}_2(F)$ , let  $A_{\mathfrak{p}}$  be the image of  $A$  in  $\text{Br}_2(F_{\mathfrak{p}})$ . So if  $u, v \in \bar{F}$ , then  $[u, v]_{\mathfrak{p}}$  is essentially the ordinary Hilbert symbol.

Before beginning the proof of (3.7) we prove a key lemma.

3.8. LEMMA. *Let  $F$  be a global field. Suppose  $A \in \text{Br}_2(F)$  is the class of an  $F$ -quaternion algebra.*

(1) Let  $S_A = \{p \in \mathbb{P}_F \mid A_p \neq 1\}$ . Then  $A \mapsto S_A$  induces a bijection between the set of quaternion algebras in  $\text{Br}_2(F)$  and the set of all  $S \subseteq \mathbb{P}_F$  with cardinality  $|S|$  finite and even.

(2) Let  $a \in \dot{F}$ . Then  $A = [a, x]$  for some  $x \in \dot{F}$  iff  $A_p = 1$  for every prime  $p \in \mathbb{P}_F$  where  $a \in \dot{F}_p^2$ .

*Proof.* (1) Hilbert reciprocity implies  $|S_A|$  is finite and even, and Hasse-Minkowski proves that  $A \rightarrow S_A$  is injective. Surjectivity follows by the global existence theorem [25, 72: 1] applied to the norm forms.

(2) Suppose  $A = [u, v]$  for some  $u, v \in \dot{F}$ . Then  $A = [a, x]$  for some  $x$  iff  $\langle a, -u, -v, uv \rangle$  is isotropic. The result then follows by Hasse-Minkowski and the observation that if a 4-dimensional quadratic form over a  $p$ -adic or real closed field is anisotropic, then its discriminant must be 1. ■

*Proof of Theorem 3.7.* By (1.21) it suffices to prove property  $Q(n)$  for all  $n \geq 2$ . Let  $n \geq 2$  be fixed and suppose  $H \subseteq \dot{F}/\dot{F}^2$  is a subgroup with  $|H| = 2^n$ . Choose a basis  $\{h_1, \dots, h_n\}$  for  $H$ . Suppose  $a, x \in \dot{F}$  satisfy  $[a, x] = \prod_{i=1}^n [h_i, y_i]$ , for some  $y_i \in \dot{F}$ , as in the hypothesis of  $Q(n)$ .

Let  $S = S_{[a, x]} = \{p \in \mathbb{P}_F \mid [a, x]_p \neq 1\}$ . Choose a prime  $q \notin S$  such that  $a, h_1, \dots, h_n \notin \dot{F}_q^2$ . Such  $q$  exists because there are infinitely many primes  $p \in \mathbb{P}_F$  with  $a, h_1, \dots, h_n \notin \dot{F}_p^2$ . This fact follows from the Global Square Theorem as in [25, 65:17]. We will hold  $q$  fixed, to be used later for adjusting parity.

*Claim.*  $[a, x] = \prod_{i=1}^n A_i$ , where  $A_i = [a, r_i] = [h_i, s_i]$ , for some  $r_i, s_i \in \dot{F}$ .

This conclusion is somewhat stronger than required for  $Q(n)$ . To find these  $A_i$ 's, we construct suitable sets  $S_i \subseteq S \cup \{q\}$ , with  $S$  being the disjoint union of the subsets  $S \cap S_i$ . For each  $p \in S$  we have  $[a, x]_p \neq 1$ , so that  $[h_i, y_i]_p \neq 1$ , for some  $i$ . Choose one such  $i$ , and put  $p$  in  $S_i$ . After distributing the elements of  $S$  among the  $S_i$ , add  $q$  to any of the  $S_i$  that would otherwise have odd cardinality. Define  $A_i$  to be the quaternion algebra corresponding to  $S_i$ , using (3.8)(1).

If  $A_{i,p} \neq 1$ , then  $p \in S_i$ , so that either  $p \in S$  or  $p = q$ . In either case,  $a, h_i \notin \dot{F}_p^2$ . Hence (4.1)(2) implies that  $A_i = [a, r_i] = [h_i, s_i]$ , for some  $r_i, s_i \in \dot{F}$ . Finally, we check locally that  $[a, x] = \prod_{i=1}^n A_i$ . This is clear by construction for every prime  $p \neq q$ , and hence it must also hold for  $q$  by Hilbert reciprocity. ■

**3.9. Remark.** A direct limit of fields satisfying  $P_1$  again has  $P_1$ . Hence, every algebraic extension field of  $\mathbb{Q}$  satisfies  $P_1$ .

An argument similar to that for global fields provides an alternative proof for fields  $F$  with  $\tilde{u}(F) \leq 2$ . The lemma corresponding to (3.8)(2) in this case says: if  $A$  is a quaternion algebra and  $a \in \dot{F}$ , then  $A = [a, x]$  for some  $x \in \dot{F}$

iff  $\text{supp}(\text{sgn } A) \subseteq H(-a)$ . Further details are left to the reader. The ideas used in the proof of (3.7) can also be used to give a somewhat different proof of the exactness theorem (3.3).

To conclude this section we mention some further results on  $P_1$  without proof.

**3.10. THEOREM.** *Suppose  $F$  is a field with a nondyadic 2-henselian valuation  $v$ , having residue field  $\bar{F}$ . Then property  $P_1(n)$  holds for  $F$  iff it holds for  $\bar{F}$ .*

The case  $v$  is a complete discrete (rank 1) valuation has been considered in [29, (1.12) and (1.13)] or [30, (6.1) and (6.2)].

**3.11. THEOREM.** *If  $F$  is a pythagorean field with finite chain length, then  $F$  satisfies  $P_1$ .*

The proof uses results of Marshall [23], Jacob [16] and (3.10) above.

#### 4. INTERSECTIONS OF 1-PFISTER IDEALS

We discuss in this section the property  $I(n)$  concerning the intersections of ideals  $\mathfrak{A}(G)$  in the Witt ring  $WF$ . The notation  $\mathfrak{A}(G)$  was introduced in (2.1). If  $a \in \dot{F}$ , we also write  $\mathfrak{A}(a) = \langle\langle -a \rangle\rangle WF$ , the ideal corresponding to the group  $H = \{1, a\} \subseteq \dot{F}/\dot{F}^2$ . In the terminology of [13], each  $\mathfrak{A}(G)$  is a 1-Pfister ideal, and conversely every 1-Pfister ideal in  $WF$  is some such  $\mathfrak{A}(G)$ .

**4.1. DEFINITION.**  $F$  has *Property  $I(n)$*  if for every subgroup  $H \subseteq \dot{F}/\dot{F}^2$  with  $|H| \leq 2^{n-1}$  and every  $a \in \dot{F}$ , then

$$\mathfrak{A}(a) \cap \mathfrak{A}(H) = \sum_{h \in H} \mathfrak{A}(a) \cap \mathfrak{A}(h).$$

This property is a Witt ring analog of property  $P_1(n)$ , using the formulation of  $P_1(n)$  in terms of property  $Q(m)$  for  $m \leq n-1$  as in (1.21). For example the reader can verify directly that if  $F$  is linked and  $I^3 F = 0$ , then  $I(n)$  and  $P_1(n)$  are equivalent. For in this case every element of  $I^2 F$  is a 2-fold Pfister form and  $c_F: I^2 F \rightarrow \text{Br}_2(F)$  is injective. (Compare (4.16) below.)

This intersection property  $I(n)$  is also considered in [13, (4.7)(3)]. It is proved there that the equality in (4.1) is true whenever  $\mathfrak{A}(H)$  is a strong 1-Pfister ideal.

In order to analyze this property  $I(n)$  more generally we introduce the complex  $\tilde{\mathcal{T}}_{M/F}^2$  formed from the ideals in the Witt rings, and the natural homomorphism  $\tilde{\mathcal{T}}_{M/F}^2 \rightarrow \mathcal{P}_{M/F}$ . Use of this new complex motivates the definition of another property  $P_1 W(n)$  which serves as a bridge between the

properties  $P_1(n)$  and  $I(n)$ . Using some local-global machinery we prove that if the Hasse number  $\tilde{u}(F) \leq 4$ , then the properties  $I(n)$ ,  $P_1 W(n)$  and  $P_1(n)$  are all equivalent. The examples found in Section 5 then provide fields where property  $I(3)$  fails. (See (4.18).)

As usual,  $WF$  denotes the Witt ring of anisotropic quadratic forms over  $F$ ,  $IF$  is the ideal of even dimensional forms,  $I^n F = (IF)^n$ , and  $\bar{I}^n F = I^n F / I^{n+1} F$ . An element of  $\bar{I}^n F$  is written  $\bar{\delta}$ , for some  $\delta \in I^n F$ . This group  $\bar{I}^n F$  is a vector space over  $\mathbb{F}_2$  (the field of 2 elements), generated by elements  $\bar{\varphi}$ , where  $\varphi$  is an  $n$ -fold Pfister form. That is,  $\varphi = \langle\langle x_1, \dots, x_n \rangle\rangle = \otimes_{i=1}^n \langle 1, x_i \rangle$ , for some  $x_i \in \dot{F}$ .

Let  $M = F(\sqrt{G})$  be a multiquadratic extension. The complex  $\bar{\mathcal{T}}_{M/F}^2$  is the following.

$$\bar{\mathcal{T}}_{M/F}^2: \bigoplus_{L \in Q(M/F)} \bar{I}^1 L \xrightarrow{\varphi''} G \otimes \bar{I}^1 F \xrightarrow{\varphi''} \bar{I}^2 F \xrightarrow{r''} \bar{I}^3 M. \quad (4.2)$$

Compare [11, (2.5)]. The map  $r''$  is given by extension of scalars. If  $a \otimes \bar{\delta} \in G \otimes \bar{I}^1 F$ , then we define  $\psi''(a \otimes \bar{\delta}) = \langle\langle -a \rangle\rangle \cdot \bar{\delta}$ . If  $(\bar{\delta}_L) \in \bigoplus \bar{I}^1 L$ , then define  $\varphi''((\bar{\delta}_L)) = \sum \kappa(L) \otimes \bar{s}_*^{L/F}(\bar{\delta}_L)$ . To complete this definition we briefly describe the transfer map  $\bar{s}_*^{L/F}$ .

For a quadratic extension  $L/F$ , choose  $s: L \rightarrow F$  to be any nonzero  $F$ -linear map. Then  $s_* = s_*^{L/F}$  denotes the associated transfer map:  $WL \rightarrow WF$ . As proved in [8, footnote 5] or [1, (2.3)], this map  $s_*$  carries  $I^n L$  into  $I^n F$ , for every  $n \geq 0$ . Let  $\bar{s}_* = \bar{s}_*^{L/F}$  be the induced map:  $\bar{I}^n L \rightarrow \bar{I}^n F$ . It is easy to check that this map  $\bar{s}_*$  does not depend on the choice of  $s$ .

If  $L = F(\sqrt{a})$ , then Frobenius reciprocity implies that  $\langle\langle -a \rangle\rangle \cdot s_*(WL) = 0$ . From this it is easy to see that  $\bar{\mathcal{T}}_{M/F}^2$  is a complex. We consider the homology groups.

**4.3. DEFINITION.** Let  $\bar{h}_1(M/F)$  and  $\bar{h}_2(M/F)$  denote the homology groups of  $\bar{\mathcal{T}}_{M/F}^2$  at  $G \otimes \bar{I}^1 F$  and  $\bar{I}^2 F$ , respectively.

It is well known [19, p. 41; 26, p. 122] that the signed discriminant  $d_+: WF \rightarrow \dot{F}/\dot{F}^2$  induces an isomorphism  $\bar{d}_+: \bar{I}^1 F \xrightarrow{\sim} \dot{F}/\dot{F}^2$ . Thus every  $\bar{\delta} \in \bar{I}^1 F$  is uniquely representable as  $\bar{\delta} = \langle\langle -d \rangle\rangle$ , for some  $d \in \dot{F}/\dot{F}^2$ .

The Witt invariant (Clifford invariant) [19, p. 120] induces a homomorphism  $\bar{c}_F: \bar{I}^2 F \rightarrow \text{Br}_2(F)$ . If  $x, y \in \dot{F}/\dot{F}^2$ , then  $\bar{c}_F$  carries the generator  $\langle\langle -x, -y \rangle\rangle$  to the quaternion algebra  $[x, y]$ . These maps  $\bar{d}_+$  and  $\bar{c}$  can be combined to form a homomorphism of complexes  $\bar{\mathcal{T}}_{M/F}^2 \rightarrow \mathcal{B}_{M/F}$  as follows.

**4.4. PROPOSITION.** For any multiquadratic extension  $M = F(\sqrt{G})$ , the following diagram is commutative.

$$\begin{array}{ccccccc}
\bigoplus_{L \in Q(M/F)} \bar{I}^1 L & \xrightarrow{\psi''} & G \otimes \bar{I}^1 F & \xrightarrow{\psi''} & \bar{I}^2 F & \xrightarrow{r''} & \bar{I}^2 M \\
\downarrow \oplus \bar{a}_{\pm} & & \downarrow 1 \otimes \bar{d}_{\pm} & & \downarrow \bar{c}_F & & \downarrow \bar{c}_M \\
\bigoplus_{L \in Q(M/F)} (\bar{L}/\bar{L}^2) & \xrightarrow{\psi} & G \otimes (\bar{F}/\bar{F}^2) & \longrightarrow & \text{Br}_2(F) & \longrightarrow & \text{Br}_2(M)
\end{array}$$

Hence there are induced maps  $\bar{\theta}_i: \bar{h}_i(M/F) \rightarrow N_i(M/F)$  for  $i = 1, 2$ .

*Proof.* Suppose  $L/F$  is a quadratic extension. Then for every  $x \in \bar{L}$  we have  $\bar{s}_*(\langle\langle -x \rangle\rangle) = \langle\langle -N_{L/F}(x) \rangle\rangle$ . This follows for instance by choosing an appropriate map  $s: L \rightarrow F$  and applying [19, pp. 195–196]. The commutativity of the leftmost square is an immediate consequence. The commutativity of the other two squares is also easy. ■

As in [11, Sect. 2] define  $\bar{I}^2 D(M/F) = \text{im } \psi'' = \{\sum_{a \in G} \langle\langle -a, -x_a \rangle\rangle \mid x_a \in \bar{F}\}$ . These are the “decomposable” elements of  $\bar{I}^2(M/F) = \ker r''$ .

4.5. PROPOSITION. Let  $M/F$  be a multiquadratic extension.

- (1)  $\bar{\theta}_1: \bar{h}_1(M/F) \rightarrow N_1(M/F)$  is injective.
- (2)  $\bar{\theta}_1$  is an isomorphism iff the map  $c_F: \bar{I}^2 F \rightarrow \text{Br}_2(F)$  is injective on  $\bar{I}^2 D(M/F)$ .
- (3) If  $[M:F] \leq 16$ , then  $\bar{\theta}_1$  is an isomorphism.

*Proof.* The left two vertical arrows in the diagram in (4.4) are isomorphisms. Then (1) follows. A diagram chase shows that  $\bar{\theta}_1$  is surjective iff  $\ker \psi''$  is carried onto  $\ker \psi$ , iff  $\bar{c}_F$  is injective when restricted to  $\text{im } \psi''$ .

For (3), assume  $M = F(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3}, \sqrt{a_4})$ . Suppose  $\bar{\xi} \in \bar{I}^2 D(M/F)$  and  $\bar{c}_F(\bar{\xi}) = 1$ . Then we may choose a representative  $\xi = \langle\langle -a_1, -x_1 \rangle\rangle - \langle\langle -a_2, -x_2 \rangle\rangle + \langle\langle -a_3, -x_3 \rangle\rangle - \langle\langle -a_4, -x_4 \rangle\rangle$ , since we are working mod  $I^3 F$ . Then the anisotropic part of  $\xi$  has dimension  $\leq 12$ , and  $c_F(\xi) = 1$  so a result of Pfister [26, Satz 14] implies  $\xi \in I^3 F$ . Then  $\bar{\xi} = 0$ , and using (2) we conclude  $\bar{\theta}_1$  is an isomorphism. ■

The following property  $P_1 W(n)$  will provide a link between the properties  $I(n)$  and  $P_1(n)$ .

4.6. DEFINITION.  $F$  satisfies Property  $P_1 W(n)$  if  $\bar{h}_1(M/F) = 0$  for every multiquadratic  $M/F$  with  $[M:F] \leq 2^n$ .

4.7. COROLLARY. (1)  $P_1(n)$  implies  $P_1 W(n)$ .

- (2) If  $\bar{c}_F$  is injective, then  $P_1(n)$  and  $P_1 W(n)$  are equivalent properties.
- (3) If  $n \leq 4$ , then  $P_1(n)$  and  $P_1 W(n)$  are equivalent.

*Proof.* Immediate from (4.5). ■

4.8. *Remark.* The theory developed for the complex  $\mathcal{B}_{M/F}$  in Section 1 can be imitated here for  $\mathcal{J}_{M/F}^2$ . If  $F \subseteq K \subseteq M$  is multiquadratic, we can define groups  $\tilde{h}_1(M/K/F)$  and get a long exact sequence like (1.9). The induced map  $\tilde{\theta}'_1: \tilde{h}_1(M/K/F) \rightarrow N_1(M/K/F)$ , together with  $\tilde{\theta}_1$  and  $\tilde{\theta}_2$ , fits into a commutative diagram relating these exact sequences. This map  $\tilde{\theta}'_1$  is always injective, and is an isomorphism if  $[M : K] \leq 16$ . Since  $M = K(\sqrt{a})$  for our applications, we may identify the groups  $\tilde{h}_1(M/K/F)$  and  $N_1(M/K/F)$ .

4.9. **PROPOSITION.** *For a field  $F$ , the following are equivalent.*

- (1)  $F$  has property  $P_1 W(n)$ .
- (2) For every subgroup  $H \subseteq \dot{F}/\dot{F}^2$  with  $|H| \leq 2^{n-1}$  and  $a, x \in \dot{F}$ , if  $\langle\langle -a, -x \rangle\rangle \in \mathfrak{A}(H) + I^3 F$ , then  $x \in \prod_{h \in H} \tilde{N}_F(ah)$ .
- (3) For every subgroup  $H \subseteq \dot{F}/\dot{F}^2$  with  $|H| \leq 2^{n-1}$  and every  $a \in \dot{F}$ ,

$$(\mathfrak{A}(a) + I^3 F) \cap (\mathfrak{A}(H) + I^3 F) = \sum_{h \in H} (\mathfrak{A}(a) \cap \mathfrak{A}(h)) + I^3 F.$$

*Proof.* The equivalence (1)  $\Leftrightarrow$  (2) is the analog of (1.21) and is proved using the same methods. To prove (2)  $\Leftrightarrow$  (3) we may assume  $a \notin H$ . If  $\xi \in (\mathfrak{A}(a) + I^3 F) \cap (\mathfrak{A}(H) + I^3 F)$ , then  $d_{\pm} \xi \in \{1, a\} \cap H$ , so that  $d_{\pm} \xi = 1$  and  $\xi \in I^2 F$ . Then since  $\xi \in \mathfrak{A}(a) + I^3 F$ , we have  $\xi \equiv \langle\langle -a, -x \rangle\rangle \pmod{I^3 F}$ , for some  $x \in \dot{F}$ . The equivalence of (2) and (3) now follows using the idea in (1.11)(2). ■

We will now investigate the properties  $I(n)$ ,  $P_1 W(n)$  and  $P_1(n)$  under the hypothesis that the field  $F$  is linked. A field  $F$  is *linked* if every pair of 2-fold Pfister forms over  $F$  is linked in the sense of [6], or equivalently, if the quaternion algebras form a subgroup of  $\text{Br}_2(F)$ .

If  $F$  is linked, then every  $q \in I^2 F$  has a unique expression  $q = \varphi + \tau$ , where  $\varphi$  is a 2-fold Pfister form and  $\tau \in I^3 F$ . (See [5, (4.4)].) This decomposition behaves well with respect to ideals:

4.10. **LEMMA.** *Suppose  $F$  is linked and  $\mathfrak{A}, \mathfrak{B}$  are 1-Pfister ideals in  $WF$ .*

- (1) *Suppose  $q \in I^2 F$  and  $q = \varphi + \tau$  as above. If  $q \in \mathfrak{A}$ , then  $\varphi, \tau \in \mathfrak{A}$ .*
- (2) *For every  $m \geq 0$ ,  $(\mathfrak{A} \cap \mathfrak{B}) + I^m F = (\mathfrak{A} + I^m F) \cap (\mathfrak{B} + I^m F)$ .*

*Proof.* (1) By [13, (2.15)],  $\mathfrak{A} \cap I^2 F = \mathfrak{A} \cdot IF$  is a 2-Pfister ideal, so that we have  $q \equiv \alpha_1 + \cdots + \alpha_r \pmod{\mathfrak{A} \cap I^3 F}$ , where each  $\alpha_i \in \mathfrak{A}$  is a 2-fold Pfister form. By the linkage argument used in the proof of [13, (2.14)], we may assume  $r = 1$ . Then  $\varphi \equiv q \equiv \alpha_1 \pmod{I^3 F}$ , so that  $\varphi = \alpha_1 \in \mathfrak{A}$ . Then also  $\tau = q - \varphi \in \mathfrak{A}$ .

(2) We may assume  $m \geq 2$  and let  $\xi \in (\mathfrak{A} + I^m F) \cap (\mathfrak{B} + I^m F)$ . Then  $\xi = \alpha + \tau_1 = \beta + \tau_2$ , where  $\alpha \in \mathfrak{A}$ ,  $\beta \in \mathfrak{B}$  and  $\tau_1, \tau_2 \in I^m F$ , so that we have  $\tau_1 - \tau_2 \in (\mathfrak{A} + \mathfrak{B}) \cap I^m F$ . Now [13, (2.15) and (2.16)] implies that  $\mathfrak{A} \cap I^m F = \mathfrak{A} \cdot I^{m-1} F$ , and similarly for  $\mathfrak{B}$ . Therefore  $(\mathfrak{A} + \mathfrak{B}) \cap I^m F = (\mathfrak{A} \cap I^m F) + (\mathfrak{B} \cap I^m F)$ , so we can express  $\tau_1 - \tau_2 = \sigma_1 + \sigma_2$  where  $\sigma_1 \in \mathfrak{A} \cap I^m F$  and  $\sigma_2 \in \mathfrak{B} \cap I^m F$ . Then  $\alpha + \sigma_1 = \beta - \sigma_2 \in \mathfrak{A} \cap \mathfrak{B}$ , and we conclude that  $\xi \in (\mathfrak{A} \cap \mathfrak{B}) + I^m F$ . The other inclusion is trivial. ■

4.11. *Remark.* By a similar proof, (4.10)(1) can be extended to any Pfister ideal in  $WF$ , for a linked field  $F$ , and any  $q \in I^n F$  with  $q = \varphi + \tau$ , for an  $n$ -fold Pfister form  $\varphi$  and  $\tau \in I^{n+1} F$ .

4.12. COROLLARY. *If  $F$  is linked, then  $I(n) \Rightarrow P_1 W(n) \Leftrightarrow P_1(n)$ .*

*Proof.* Since  $F$  is linked,  $\bar{c}_F$  is injective and the equivalence of  $P_1 W(n)$  and  $P_1(n)$  follows from (4.7). The fact that  $I(n)$  implies  $P_1 W(n)$  is a consequence of (4.10)(2) applied to  $\mathfrak{A} = \mathfrak{A}(a)$ ,  $\mathfrak{B} = \mathfrak{A}(H)$  and  $m = 3$ . ■

In order to get a converse to the implication in (4.12) we investigate some analogous properties for the rings of continuous functions. When  $\bar{u}(F) \leq 4$  we will be able to pull back these properties to show that  $I(n)$  and  $P_1 W(n)$  are equivalent.

Let  $C(F) = \mathcal{C}(X_F, \mathbb{Z})$  be the ring of continuous functions, so that  $C(F)/2C(F) \cong \bar{C}(F)$ , the reduced ring examined in Section 3. The signature map  $\text{sgn}: WF \rightarrow C(F)$  carries  $I^n F$  into  $2^n C(F) = \mathcal{C}(X_F, 2^n \mathbb{Z})$ . If  $S \subseteq X_F$  is clopen, we can identify the ideal  $\chi_S \cdot C(F)$  with the ring  $\mathcal{C}(S, \mathbb{Z})$ . Here  $\chi_S$  is the characteristic function of the set  $S$ . Let us use the notation

$$C(S) = \chi_S \cdot C(F) = \{f \in C(F) \mid \text{supp } f \subseteq S\}.$$

Here, as in [11, Sect. 6], if  $f \in C(F)$ , then the support of  $f$  is  $\text{supp } f = \{\alpha \in X_F \mid f(\alpha) \neq 0\}$ . If  $\alpha \in WF$ , define  $\text{supp } \alpha = \text{supp}(\text{sgn } \alpha)$ .

4.13. LEMMA. *If  $S, T \subseteq X_F$  are clopen, then  $C(S \cap T) = C(S) \cap C(T)$  and  $C(S \cup T) = C(S) + C(T)$ , as ideals in  $C(F)$ .*

*Proof.* Clear. ■

4.14. *Remark.* This lemma can be interpreted as saying that intersections of Pfister ideals in  $C(F)$  are again Pfister ideals. Here a "Pfister ideal" in  $C(F)$  is an ideal generated by functions  $\chi_S$  for clopen sets  $S$ . That is, if  $\{S_i\}$  is a family of clopen sets in  $X_F$ , then  $\sum C(S_i)$  is a Pfister ideal in  $C(F)$ . The connection with Pfister forms in  $WF$  is given as follows: (1) If  $S = H(a_1, \dots, a_n)$ , then  $\chi_S = 2^{-n} \cdot \text{sgn} \langle a_1, \dots, a_n \rangle$ . (2) Every clopen set  $S$  in  $X_F$  is a finite union of such basic sets  $H(a_1, \dots, a_n)$ .



A field  $F$  is *reduced  $n$ -stable* [11, (8.1)] if  $I^{n+1}F \subseteq I^n F + W_i(F)$ . Equivalently [11, (8.3)],  $F$  is reduced  $n$ -stable iff the map  $\text{sgn}: I^n F \rightarrow 2^n C(F)$  is surjective. A basic result of Pfister [19, p. 240] says that the kernel of this map is the torsion part of  $I^n F$ . Hence if  $I^n F$  is torsion free, the map is injective.

**4.15. PROPOSITION.** *Suppose  $F$  is reduced  $(n-1)$ -stable and  $I^n F$  is torsion free, for some  $n \geq 1$ . Let  $G \subseteq \dot{F}/\dot{F}^2$  be a subgroup and  $a \in \dot{F}$ . Then*

$$\begin{aligned} \mathfrak{A}(a) \cap \mathfrak{A}(G) \cap I^n F &= \left( \sum_g \mathfrak{A}(a) \cap \mathfrak{A}(g) \right) \cap I^n F \\ &= \sum_g (\mathfrak{A}(a) \cap \mathfrak{A}(g) \cap I^n F). \end{aligned}$$

*Proof.* Since  $I^n F$  is torsion free, it maps injectively to  $2^n C(F)$ . Thus, it suffices to see that the ideals in the proposition have the same image in  $2^n C(F)$ . Note that  $\text{sgn}(\mathfrak{A}(b) \cap I^n F) = 2^n C(H(-b))$ , for any  $b \in \dot{F}$ . For, since  $F$  is reduced  $(n-1)$ -stable, if  $f \in 2^n C(H(-b))$ , we may choose  $\beta \in I^{n-1}F$  with  $2 \text{sgn } \beta = f$ . Then  $f = \text{sgn}(\langle\langle -b \rangle\rangle \beta)$ . It follows from the injectivity of  $\text{sgn}$  that  $\text{sgn}(\mathfrak{A}(a) \cap \mathfrak{A}(g) \cap I^n F) = 2^n [C(H(-a)) \cap C(H(-g))]$ . Using this and (4.13), we have

$$\begin{aligned} \text{sgn}(\mathfrak{A}(a) \cap \mathfrak{A}(G) \cap I^n F) &\subseteq 2^n C(H(-a)) \cap C \left( \bigcup_{g \in G} H(-g) \right) \\ &= \sum_g 2^n [C(H(-a)) \cap C(H(-g))] \\ &= \text{sgn} \left( \sum_g (\mathfrak{A}(a) \cap \mathfrak{A}(g) \cap I^n F) \right). \end{aligned}$$

Since the inclusions  $\supseteq$  for the ideals in (4.15) are clear, this completes the proof. ■

**4.16. THEOREM.** *If  $\tilde{u}(F) \leq 4$ , then for every  $n \geq 1$ , the properties  $I(n)$ ,  $P_1 W(n)$  and  $P_1(n)$  are equivalent.*

*Proof.* By [5, 4.7)],  $\tilde{u}(F) \leq 4$  iff  $F$  is linked and  $I^3 F$  is torsion free. Every field with  $\tilde{u}(F) < \infty$  has SAP, i.e., is reduced 1-stable, and hence (4.15) applies with  $n = 3$ . By (4.12) it remains to prove that  $P_1 W(m)$  implies  $I(m)$  for every  $m$ . Suppose  $H \subseteq \dot{F}/\dot{F}^2$  is a subgroup with  $|H| \leq 2^{m-1}$  and  $a \in \dot{F}$ , and let  $\mathfrak{A} = \mathfrak{A}(a) \cap \mathfrak{A}(H)$  and  $\mathfrak{B} = \sum_h \mathfrak{A}(a) \cap \mathfrak{A}(h)$ . Then  $\mathfrak{A} \subseteq \mathfrak{B}$  and  $P_1 W(m)$  implies that  $\mathfrak{A} + I^3 F = \mathfrak{B} + I^3 F$ . On the other hand, (4.15) implies that  $\mathfrak{A} \cap I^3 F = \mathfrak{B} \cap I^3 F$ . It quickly follows that  $\mathfrak{A} = \mathfrak{B}$ , so that property  $I(m)$  is verified. ■

4.17. *Remark.* If  $\tilde{u}(F) \leq 2$ , then this theorem and (3.6) imply that  $F$  has  $I(n)$  for all  $n$ . In fact the machinery in this section can be used to provide a new proof of (3.6).

4.18. *Remark.* Theorems (4.16) and (3.7) show that every global field has property  $I(n)$  for all  $n$ . However for fields  $F$  with  $\tilde{u}(F) \leq 4$  which do not have  $P_1(3)$ , then also property  $I(3)$  fails. For example  $F = \mathbb{C}(x, y)$  is such a field, where  $\mathbb{C}$  is the field of complex numbers. For by Tsen-Lang [19, p. 296] we have  $\tilde{u}(F) = u(F) = 4$ , and by (5.10) below,  $P_1(3)$  fails for  $F$ .

## 5. COUNTEREXAMPLES TO $P_1(3)$

The second author proved in [29, Ch. 4] that  $\mathbb{Q}(x, y)$  does not satisfy property  $P_1(3)$ . We generalize that example here, always using the idea of dyadic valuations.

For valuations we generally follow the notation of [4]. If  $v$  is a valuation on a field  $F$ , then  $A(v)$ ,  $\mathfrak{m}(v)$ ,  $U(v)$ ,  $\Gamma(v)$  and  $k(v)$  denote the valuation ring, maximal ideal, group of units, value group, and residue field, respectively. Here  $\Gamma(v)$  is an *ordered group*, that is, a totally ordered additive abelian group. The valuation  $v$  is *dyadic* if  $v(2) > 0$ , that is, if the residue field  $k(v)$  has characteristic 2.

If  $k$  is a field of characteristic 2, define  $\wp(k) = \{a^2 + a \mid a \in k\}$ . Then  $k^2$  and  $\wp(k)$  are additive subgroups of  $k$ . Note that  $k$  has a separable quadratic extension iff  $k \neq \wp(k)$ , and  $k$  has an inseparable quadratic extension iff  $k \neq k^2$ .

5.1. THEOREM. *Suppose  $F$  is a field of characteristic 0 with a dyadic valuation  $v$  having  $k(v) \neq k(v)^2$ . Suppose further that either*

- (i) *there exists  $\pi \in \dot{F}$  with  $0 < v(\pi) < v(2)$  and  $v(\pi) \notin 2\Gamma(v)$ ; or*
- (ii)  *$v(2) \notin 2\Gamma(v)$  and  $k(v) \neq \wp(k(v))$ .*

*The  $F$  does not satisfy property  $P_1(3)$ .*

*Proof.* To shorten notation we write  $A$ ,  $\mathfrak{m}$ ,  $U$ ,  $\Gamma$  and  $k$  for the various objects associated to  $v$ . In case (ii), let  $\pi = 2$ . Then  $v(\pi) \notin 2\Gamma$  in either case. Choose  $c \in U$  such that  $\bar{c} \in k - k^2$ , and, in case (ii), such that  $\bar{c}^{-1} \notin \wp(k)$ . Such  $c$  exists since, in case (ii),  $k \neq k^2 \cup \wp(k)$  (a group is not the union of two proper subgroups).

Let  $a = c + \pi^2$  and  $b = c + 1$ . We will prove that  $N_1(a, b, c) \neq 1$ . To do this, let  $E = F(\sqrt{c})$  and note that  $\pi \in \dot{F} \cap N_E(a)N_E(b)$ . In fact,  $\pi = (2\pi\sqrt{c})(2\sqrt{c})^{-1}$  where  $2\pi\sqrt{c} = (\sqrt{c} + \pi)^2 - a \in N_E(a)$  and  $2\sqrt{c} = (\sqrt{c} + 1)^2 - b \in N_E(b)$ . Therefore, by (1.15) it suffices to prove

$\pi \notin (\dot{F} \cap N_E(a))(\dot{F} \cap N_E(b))$ . Equivalently, by (1.14), we need:  $\pi \notin N_F(a) N_F(ac) N_F(b) N_F(bc)$ . Since  $v(\pi) \notin 2\Gamma$ , it suffices to show that  $v$  maps each of these four norm groups into  $2\Gamma$ .

Since  $\bar{c} \notin k^2$ , we have  $\bar{a} = \bar{c}$ ,  $\bar{b} = \bar{c} + 1$ , and  $\overline{bc} = \bar{c}^2 + \bar{c}$  are not in  $k^2$ . Hence, [11, (5.5)(a)] shows that  $v(N_F(a)) \subseteq 2\Gamma$ , and likewise for  $b$  and  $bc$ .

The argument for  $ac$  is more delicate. We first claim that  $ac \notin \dot{F}^2$ . Suppose not. Then there is a  $d \in \dot{F}$ , with  $d^2 = ac = c^2 + c\pi^2$ . Let  $r = d - c$ . Then, from  $c\pi^2 = d^2 - c^2 = (c + r)^2 - c^2$ , we have

$$c\pi^2 = 2rc + r^2. \quad (1)$$

Suppose momentarily that  $v(r^2) < v(2rc)$ . Then (1) shows that  $v(r) = v(\pi)$  and  $\bar{c} = (r/\pi)^2 \in k^2$ , contradicting the choice of  $c$ . Hence,  $v(r^2) \geq v(2rc)$ , i.e.,  $v(r) \geq v(2)$ . But then (1) shows

$$v(c\pi^2) \geq v(2rc) = v(2) + v(r) \geq 2v(2). \quad (2)$$

Thus,  $v(\pi) \geq v(2)$ . But in case (i) of the theorem,  $v(\pi) < v(2)$ , a contradiction. In case (ii),  $\pi = 2$ , so the inequalities in (2) must be equalities, forcing  $v(r) = v(2) = v(\pi)$ . Then, solving (1) for  $c^{-1}$  (with  $\pi = 2$ ) yields  $c^{-1} = (\pi/r)^2 - (\pi/r)$ . Hence,  $\bar{c}^{-1} \in \wp(k)$ , contradicting the choice of  $c$ . Since either case leads to a contradiction, we must have  $ac \notin \dot{F}^2$ , as claimed.

Let  $K = F(\sqrt{ac})$ , and let  $w$  be an extension of  $v$  to  $K$ . Observe that if  $k(w) = k$ , then the argument for the claim could be repeated with  $K$  and  $w$  replacing  $F$  and  $v$ , leading to the absurdity that  $ac \notin F(\sqrt{ac})^2$ . Hence,  $k(w) \neq k$  and [11, (5.5)(a)] applies again, showing that  $v(N_F(ac)) \subseteq 2\Gamma$ , and completing the proof. ■

The theorem becomes a little simpler when the valuation is discrete (i.e.,  $\Gamma \cong \mathbb{Z}$ ). In this situation we may assume  $\Gamma = \mathbb{Z}$ .

**5.2. COROLLARY.** *Suppose  $F$  has characteristic 0 and has a discrete dyadic valuation with residue field  $k$ . If  $k \neq k^2$  and if  $k \neq \wp(k)$  or  $v(2) \geq 2$ , then  $F$  does not satisfy  $P_1(3)$ .*

*Proof.* With the identification  $\Gamma = \mathbb{Z}$ , choose  $\pi \in \dot{F}$  with  $v(\pi) = 1$ . Then  $v(\pi) \notin 2\Gamma$  and  $0 < v(\pi) \leq v(2)$ . If  $v(2) > 1$  we have case (i) of (5.1), while if  $v(2) = 1$ , we have case (ii). ■

**5.3. COROLLARY.**  *$\mathbb{Q}(x)$  does not satisfy  $P_1(3)$ .*

*Proof.* Let  $v$  be the discrete valuation whose ring is  $\mathbb{Z}[x]$  localized at the prime ideal (2). Then  $k(v) \cong \mathbb{F}_2(x)$ , where  $\mathbb{F}_2$  is the field with two elements. Since  $x \in k(v)$  is not in  $k(v)^2$  nor in  $\wp(k(v))$ , the result follows from (5.2). ■

5.4. *Remark.* Following through the proof of (5.1) in the case  $F = \mathbb{Q}(x)$ , we can choose  $\pi = 2$  and  $c = x$ . Therefore  $N_1(x+4, x+1, x) \neq 1$ . Consequently, from (2.9) we see that property CV fails for  $E/F$  where  $E = F(\sqrt{x})$ ,  $\alpha = \langle 1, -(x+4) \rangle$  and  $\beta = 2\langle 1, -(x+1) \rangle$ .

We see from the next theorem and its corollaries that there are large classes of fields not satisfying  $P_1(3)$ . The proofs of (5.5)–(5.7) will occupy the rest of this section.

5.5. **THEOREM.** *Let  $L$  be a field of characteristic 0 with a dyadic valuation  $v$ . Suppose either (i)  $k(v) \neq k(v)^2$ , or (ii) there is a  $\pi \in L$  with  $0 < v(\pi) \leq v(2)$  and  $v(\pi) \notin 2\Gamma$ . If  $F$  is a finitely generated but not algebraic extension of  $L$ , then  $F$  does not satisfy  $P_1(3)$ .*

5.6. **COROLLARY.** *Suppose  $L$  is a field of characteristic 0 and  $F/L$  is a finitely generated extension of transcendence degree  $\geq 2$ . Then  $F$  does not satisfy  $P_1(3)$ .*

5.7. **COROLLARY.** (a) *Suppose  $F$  is finitely generated over  $\mathbb{Q}$ . Then  $F$  satisfies  $P_1(3)$  iff  $F$  is algebraic over  $\mathbb{Q}$ .*

(b) *Suppose  $F$  is finitely generated over  $R$ , where  $R$  is a real closed field. Then  $F$  satisfies  $P_1(3)$  iff  $\text{tr deg}(F/R) \leq 1$ .*

Before proving (5.5) some preliminary results are needed. After first dealing with simple transcendental extensions, we will consider properties inherited under finitely generated extensions.

5.8. **LEMMA.** *Suppose  $L$  is a field with valuation  $v$ , and  $F = L(x)$  is a rational function field. Then  $v$  extends to a valuation  $w$  of  $F$  having  $\Gamma(w) = \Gamma(v)$  and  $k(w) \cong k(v)(x)$ , the rational function field over  $k(v)$ . So,  $k(w) \neq k(w)^2$  and  $k(w) \neq \wp(k(w))$ .*

*Proof.* This is very well known—see, e.g., [4, Sect. 10, n° 1, Proposition 2]. The valuation  $w$  is defined by: if  $f(x) = \sum a_i x^i \in L[x]$ , then  $w(f(x)) = \min\{v(a_i)\}$ . Clearly (the image of)  $x$  in  $k(w)$  does not lie in  $k(w)^2$  nor in  $\wp(k(w))$ . ■

5.9. **PROPOSITION.** *Suppose  $L$  is a field of characteristic 0 having a dyadic valuation  $w$  with residue field  $k$ , such that  $k \neq k^2$ . Then  $F = L(x)$  satisfies the hypotheses of (5.1)(i).*

*Proof.* Let  $v$  be an extension of  $w$  to  $L(x)$  as described in (5.8) (with the roles of  $v$  and  $w$  reversed). The residue field  $k(w)$  then has a discrete valuation  $v_0$  with residue field  $k$ . (Use the  $x$ -adic valuation.)

Let  $v'$  be the "composite" of the valuations  $v$  and  $v_0$ , in the sense of [32, p. 43]. We will describe  $v'$  more explicitly. For  $f \in A(v)$  let  $\bar{f}$  denote the image of  $f$  in the residue field  $k(x)$ . Define  $A = \{f \in A(v) \mid \bar{f} \in A(v_0)\}$ . It follows that  $A$  is the valuation ring of a valuation  $v'$  on  $F$ , and the residue field of this composite valuation is  $k(v') \cong k$ .

Choose any  $\pi \in U(v)$  such that  $0 < v_0(\pi) \notin 2\mathbb{Z}$ . Then  $\pi \in \mathfrak{m}(v')$  and  $2/\pi \in \mathfrak{m}(v) \subseteq \mathfrak{m}(v')$ . Therefore  $0 < v'(\pi) < v'(2)$ . To verify (5.1)(i) it remains to show that  $v'(\pi) \notin 2\Gamma(v')$ . If not, then  $\pi A = p^2 A$  for some  $p \in A$ . But then  $\bar{\pi}A(v_0) = \bar{p}^2 A(v_0)$  and we get  $v_0(\pi) = 2v_0(\bar{p}) \in 2\mathbb{Z}$ , contradicting the choice of  $\pi$ . ■

5.10. *Remark.* We can see already that if  $F = L(x, y)$  is purely transcendental over a field  $L$  of characteristic 0, then  $F$  satisfies the hypotheses of (5.1), so  $F$  does not satisfy  $P_1(3)$ . For, starting with any dyadic valuation on  $L$ , first apply (5.8) to the extension  $L(x)/L$ , then (5.9) to  $F/L(x)$ . Specifically, we have  $N_1(x + y^2, x + 1, x) \neq 1$ , using  $c = x$  and  $\pi = y$ .

To deal with algebraic extensions we need two lemmas.

5.11. **LEMMA.** *Suppose  $k$  is a field of characteristic 2, and  $E/k$  is a finite algebraic extension.*

- (1) *If  $k \neq k^2$ , then  $E \neq E^2$ .*
- (2) *If  $k \neq \wp(k)$ , then  $E \neq \wp(E)$ .*

*Proof.* (1) Suppose  $k \neq k^2$  but  $E = E^2$ . Then  $k \subset k^{1/2} \subset \dots \subset k^{1/2^n} \subset \dots \subset E$ . Each step is proper, since the square map  $x \rightarrow x^2$  is injective. Hence  $[E : k]$  must be infinite.

(2) Suppose  $k \neq \wp(k)$  but  $E = \wp(E)$ . For  $a \in E$  let  $\wp^{-1}a$  denote a root of  $X^2 + X + a$ . Then  $\wp^{-1}a \in E$ , by hypothesis. Choose  $a_1 \in k - \wp(k)$  and let  $k_1 = k(\wp^{-1}a_1)$ . Let  $a_2 = a_1 \wp^{-1}a_1 \in k_1$  and note that  $a_2 \notin \wp(k_1)$ . Let  $k_2 = k_1(\wp^{-1}a_2)$ , and continue the process to obtain  $k \subset k_1 \subset k_2 \subset \dots \subset E$ . Each step is a quadratic extension, forcing  $[E : k]$  to be infinite. ■

Suppose  $H \subseteq G$  are ordered groups. If the index  $(G : H)$  is finite, then  $H \neq 2H$  implies  $G \neq 2G$ . This fact is sharpened in the next lemma, following the idea in [4, Sect. 8, n° 4, Proposition 3].

5.12. **LEMMA.** *Suppose  $H \subseteq G$  are ordered groups with  $(G : H) < \infty$ .*

- (a) *If  $H$  has a minimal positive element, then so does  $G$ .*
- (b) *If  $h \in H - 2H$ , then there exists an integer  $n \geq 0$  with  $g = 2^{-n}h \in G$  but  $g \notin 2G$ .*

*Proof.* (a) Suppose  $x \in H$  is a minimal positive element of  $H$ , and suppose  $G$  has no minimal positive element. Then there exist  $y_i \in G$  with  $x > y_1 > y_2 > \dots > 0$ . Since  $(G : H) < \infty$ , some of these  $y_i$  are congruent mod  $H$ . Hence, we have  $0 < y < y' < x$ , for some  $y, y' \in G$  with  $y' - y \in H$ . But then  $0 < y' - y < x$ , contrary to the minimality of  $x$ .

(b) Given  $h \in H - 2H$ , if the conclusion fails, then  $2^{-n}h \in G$  for all  $n \in \mathbb{N}$ . Let  $G_0$  be the subgroup of  $G$  generated by all these  $2^{-n}h$ .

*Claim.*  $G_0 \cap H = \mathbb{Z}h$ .

If so, then  $G_0$  has no minimal positive element but  $G_0 \cap H$  does have one. This contradicts part (a), because  $(G_0 : G_0 \cap H) = (G_0 H : H) \leq (G : H) < \infty$ .

To prove the claim, suppose  $x \in G_0 \cap H$  but  $x \notin \mathbb{Z}h$ . Then  $x = m2^{-n} \cdot h$ , where  $m$  is odd and  $n \geq 1$ . Choosing  $m' \in \mathbb{Z}$  with  $m'm \equiv 1 \pmod{2^n}$ , we find  $2^{-n}h \equiv m'x \equiv 0 \pmod{H}$ , contrary to the choice  $h \notin 2H$ . ■

**5.13. PROPOSITION.** *Suppose  $K$  is a finitely generated extension of a field  $F$  of characteristic 0. Let  $v$  be a dyadic valuation on  $F$  having any of the following properties:*

- (i)  $k(v) \neq k(v)^2$ ;
- (ii)  $k(v) \neq \wp(k(v))$ ;
- (iii) *there is a  $\pi \in \dot{F}$  with  $0 < v(\pi) < v(2)$  and  $v(\pi) \notin 2\Gamma$ ;*
- (iv) *there is a  $\pi \in \dot{F}$  with  $0 < v(\pi) \leq v(2)$  and  $v(\pi) \notin 2\Gamma$ .*

*Then there is an extension  $w$  of  $v$  to  $K$ , such that  $w$  has each of properties (i)–(iv) that  $v$  has. Hence, if  $F$  satisfies the hypotheses of (5.1), then so also does  $K$ .*

*Proof.* It suffices to prove the proposition for  $K$  a simple extension of  $F$ . First, suppose  $K = F(x)$ , where  $x$  is transcendental over  $F$ . Let  $w$  be an extension of  $v$  to  $K$  as described in (5.8). Then (i) and (ii) hold for  $w$ . Also, since  $\Gamma(w) = \Gamma(v)$ , (iii) or (iv) will hold for  $w$  if it holds for  $v$ , with the same choice of  $\pi$ .

Now, suppose  $K$  is algebraic over  $F$ . Let  $w$  be any extension of  $v$  to  $K$ . We have the well-known inequality  $(\Gamma(w) : \Gamma(v))[k(w) : k(v)] \leq [K : F] < \infty$ , as in [4, Sect. 8, n° 1, Lemma 2]. Thus, (5.11) shows that if  $k(v)$  has property (i) or (ii), then so has  $k(w)$ . Suppose now that  $v$  has property (iii) (resp. (iv)). Lemma 5.12(b) applies with  $G = \Gamma(w)$ ,  $H = \Gamma(v)$ , and  $h = v(\pi)$ . Thus, there is an integer  $n \geq 0$  and a  $\pi' \in \dot{K}$  with  $w(\pi') \notin 2\Gamma(w)$  and  $2^n w(\pi') = v(\pi)$ . Hence,  $w(\pi') > 0$ , so  $w(\pi') \leq 2^n w(\pi') = v(\pi)$ . Since  $v(\pi) < v(2) = w(2)$  (resp.  $v(\pi) \leq w(2)$ ), we have  $w(\pi') < w(2)$  (resp.  $w(\pi') \leq w(2)$ ), i.e.,  $w$  has property (iii) (resp. (iv)).

It follows immediately that if  $F$  satisfies the hypotheses of (5.1), then so does  $K$ . (Note, though, that if  $(F, v)$  is in case (ii) of (5.1),  $(K, w)$  may end up in case (i) instead of case (ii).) ■

It is now easy to prove (5.5)–(5.7).

*Proof of (5.5).* Choose any  $x$  in  $F$  transcendental over  $L$ . In case (i) of (5.5), Proposition 5.9 says that  $L(x)$  satisfies the hypotheses of (5.1). In case (ii) of (5.5), let  $w$  be an extension of  $v$  to  $L(x)$  as described in (5.8). Then, since  $\Gamma(w) = \Gamma(v)$ , hypothesis (ii) shows that  $L(x)$  satisfies the hypotheses of (5.1). The theorem thus follows in either case by applying (5.13) to the extension  $F/L(x)$ . ■

*Proof of (5.6).* By [4, Sect. 2, n° 4], the 2-adic valuation of  $\mathbb{Q}$  extends to a valuation  $v$  of  $L$ . Take any  $x$  in  $F$  transcendental over  $L$ , and extend  $v$  to a valuation  $w$  of  $L(x)$  as in (5.8). Then  $k(w) \neq k(w)^2$ , and we can apply (5.5) to the extension  $F/L(x)$ . ■

*Proof of (5.7).* (a) follows from (3.7) and (5.5)(ii) with  $L = \mathbb{Q}$ . Similarly, (3.6) and (5.6) imply (b), since if  $\text{tr deg}(F/R) \leq 1$ , then by [10, Theorem I],  $u(F) \leq 2$ . ■

5.14. *Remarks.* (i) One can construct examples of a field of characteristic 0 with a dyadic valuation  $v$  for which  $\Gamma(v)$  is not 2-divisible but neither (i) nor (ii) of (5.1) hold for  $\Gamma(v)$ . For example, one can arrange  $\Gamma(v) \cong \mathbb{Z} \times \mathbb{Q}$ , ordered lexicographically, with  $v(2) = (0, 1)$ .

(ii) If  $(F, v)$  meets the conditions of (5.1)(ii) but not (5.1)(i), then  $v(2)$  is the minimal positive element of  $\Gamma(v)$ . That is 2, generates the maximal ideal  $\mathfrak{m}(v)$ .

(iii) The value group of the valuation  $v'$  on  $L(x)$  in the proof of (5.9) is isomorphic to  $\Gamma(w) \times \mathbb{Z}$  ordered lexicographically. In this isomorphism  $v'(2)$  corresponds to  $(w(2), 0)$  and  $v'(\pi)$  to  $(0, n)$  for  $n > 0$ ,  $n$  odd.

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*Note added in proof.* It has recently been proved by A. S. Merkurjev that the map  $g_F: k_2 F \rightarrow \text{Br}_2(F)$  is an isomorphism, for every field  $F$  of characteristic  $\neq 2$ . Hence the complex  $\mathcal{B}_F$  of (1.3) is always exact (even though the relative complex  $\mathcal{B}_{M/F}$  sometimes is not) and the properties  $P_1 W(n)$  and  $P_1(n)$  of Section 4 are equivalent (see 4.7(2)).

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